ORBITS OF SU(2) - REPRESENTATIONS AND MINIMAL ISOMETRIC IMMERSIONS

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Abstract. In contrast to all known examples, we show that in the case of minimal isometric immersions of S^3 into S^N the smallest target dimension is almost never achieved by an $SU(2)$–equivariant immersion. We also give new criteria for linear rigidity of a fixed minimal isometric immersion of S^3 into S^N.

The minimal isometric immersions arising from irreducible $SU(2)$–representations are linearly rigid within the moduli space of $SU(2)$–equivariant immersions. Hence the question arose whether they are still linearly rigid within the full moduli space. We show that this is false by using our new criteria to construct an explicit $SU(2)$-equivariant immersion which is not linearly rigid.

Various authors [GT], [To3], [W1] have shown that minimal isometric immersions of higher isotropy order $d \geq 1$ play an important role in the study of the moduli space of all minimal isometric immersions of S^3 into S^N. Using a new necessary and sufficient condition for immersions of isotropy order $d \geq 1$, we derive a general existence theorem of such immersions.

0. Introduction

For more than 30 years minimal isometric immersions of Riemannian manifolds into round spheres have been extensively studied ([C], [DW2], [L], [T], [TZ]). Naturally arising questions included: What are necessary and sufficient conditions for the existence of such immersions? Exactly which manifolds admit such immersions? Do these immersions have any kind of uniqueness properties? What are the possible codimensions of these immersions, in particular what are the smallest codimensions of such immersions? The first two questions both deal with existence only; for answers, see [DZ], [E1], [E2]. This paper addresses the last two questions, whose answers provide more insight into the structure of the moduli space of all minimal isometric immersions.

A necessary and sufficient condition for the minimality of isometric immersions was given by T. Takahashi [T] who observed that if $F\colon M \longrightarrow S^N(r) \subset \mathbb{R}^{N+1}$ is a minimal isometric immersion, then all components of F must be eigenfunctions

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of the Laplace operator on M with respect to the same eigenvalue. Conversely if F is an isometric immersion such that all its components are eigenfunctions of the Laplace operator for the same eigenvalue, then F is a *minimal* isometric immersion into a round sphere.

Takahashi also observed that if M is an isotropy irreducible Riemannian homogeneous space, i. e. if the isotropy group of a point acts irreducibly on the tangent space, then an orthonormal basis of each eigenspace automatically gives rise to a minimal isometric immersion into a round sphere. We call these the *standard* minimal immersions.

When $M = S^n(1)$, there is a sequence of standard minimal isometric immersions, one for each nonzero eigenvalue k. In this case the eigenfunctions are harmonic homogeneous polynomials of degree k. For the first such eigenvalue one obtains the standard embedding into $\mathbb{R}^{n+1}$, for the second eigenvalue an immersion into $S^{\frac{n(n+3)}{2}-1}\left(\sqrt{\frac{n}{2(n+1)}}\right)$, which gives rise to the Veronese embedding of $\mathbb{R}P^n$. It turns out that for odd–numbered eigenvalues the images are always embedded spheres and for even–numbered eigenvalues the images are always embedded real projective spaces, see [DZ] for more details.

The first uniqueness result was obtained by E. Calabi [C], who showed that *every* minimal isometric immersion of $S^2(1)$ into $S^N(r)$ is congruent to one of the standard eigenspace immersions. Here two immersions are called *congruent* if they differ by an isometry of the ambient space, in this case by an element of $O(N+1)$, the isometry group of $S^N(r)$. However, M. do Carmo and N. Wallach [DW2] showed that in higher dimensions there are in general many minimal isometric immersions of $S^n(1)$ into $S^N(r)$ and that they are parametrized by a compact convex body in a finite dimensional vector space. A recent development is the determination of the exact dimension of this convex body by G. Toth [To1]. For further references on this matter, see [DW1], [DW2], [L], [T].

A fundamental problem posed by do Carmo and Wallach [DW2], [Wa] is the question of the smallest dimension of the target sphere $S^N(r)$, when the domain dimension n and the degree k are fixed.

Definition 0.1. *Fix natural numbers $n \geq 3$ and $k > 0$ and consider the set of all minimal isometric immersions $F : S^n \longrightarrow S^N$ of degree k. We call the* **minimal target dimension**, *$N(n,k)$, the smallest possible value of the dimension N of the various target spheres.*

One of our main tools is a specific rigidity property of minimal isometric immersions. We call an immersion $F : S^n \longrightarrow S^N$ *linearly rigid* if for all linear $A \in M(N+1,\mathbb{R})$ such that $A \circ F : S^n \longrightarrow \mathbb{R}^{N+1}$ is an isometry into the unit sphere S^N, $A \circ F$ is congruent to F.

Note that the space of congruence classes of minimal isometric immersions of the form $A \circ F$ for $A \in M(N+1,\mathbb{R})$ is again a compact convex body B_F. If the immersion F is not linearly rigid, i. e. if there is $F' = A \circ F$ not congruent to F, then we can deform F into F' and continue to deform further until we reach a boundary point of B_F, necessarily mapping into a sphere of dimension less than N. Summarizing this argument we obtain the following well known result.

Fact. *The minimal target dimension can only be realized by extremal points in the compact convex body parametrizing all minimal isometric immersions. These extremal points are precisely the points corresponding to linearly rigid minimal isometric immersions. Hence, if a minimal isometric immersion* $F : S^n \longrightarrow S^N$ *of degree* k *is not linearly rigid, then there exists a minimal isometric immersion of smaller target dimension, i. e. there exists a minimal isometric immersion* $\tilde{F} : S^n \longrightarrow S^{\tilde{N}}$ *of degree* k *with strictly smaller target dimension* $\tilde{N} < N$ *. In particular,* F *does not realize the minimal target dimension* $N(n,k)$ *for this* n *and* k *.*

Except for the minimal isometric immersions of inhomogeneous lens spaces into spheres as studied in [E2], all explicit examples of minimal isometric immersions in the literature are $SU(2)$–equivariant. Moreover, one knows that within the class of $SU(2)$–equivariant minimal isometric immersions from S^3 into S^N of even degree $k \geq 6$ the minimal target dimension is $N(3,k)^{SU(2)} = k$ and for odd degree $k \geq 5$ it is $N(3,k)^{SU(2)} = 2k+1$ [Ma1]. Consequently, all examples of such $SU(2)$–equivariant immersions of target dimension $N(3,k)^{SU(2)}$ are linearly rigid among the $SU(2)$–equivariant immersions. The question arose whether this restriction can be removed to conclude that these immersions are linearly rigid within the full moduli space. We show, by construction, that this is not possible.

Theorem 1. *There exists an* $SU(2)$*–equivariant minimal isometric immersion* $S^3 \longrightarrow S^{36}$ *of degree* 36 *which is not linearly rigid. Furthermore, this immersion admits at least a nine-dimensional space of deformations.*

In examples known to date, $SU(2)$–equivariance was key to obtaining remarkably small target dimensions [Ma1], [DZ]. Indeed, in the particular case of $k = 4$ it was shown that $SU(2)$–equivariant immersions minimize target dimensions [TZ]. One might surmise that $SU(2)$–equivariant immersions always minimize target dimension. However, that is not so, as we show.

Corollary 1.1. *For* $k = 36$ *, no* $SU(2)$*–equivariant minimal isometric immersion achieves the minimal target dimension.*

In the recent past several authors [GT], [To3], [W1] have noticed the importance of imposing additional constraints on the class of minimal isometric immersions in order to analyze the moduli space of all such immersions further. A natural constraint is to replace the differential F^* of an immersion $F : S^n \longrightarrow S^N$ in the isometry condition by the higher fundamental forms $\mathbb{I}^b(F)$, $b = 0, \ldots, d$. An immersion F satisfying these $d+1$ equations is called of "isotropy order d".

Example. *The minimal isometric immersion* $S^3 \longrightarrow S^{12}$ *of degree* 12 *first constructed in* [DZ] *was proven to be of isotropy order* 2 *in* [W1]. *In odd degrees one usually expects large target dimensions. Indeed, one can show that in odd degrees strictly less than* 7 *all isotropy order* 2 *minimal isometric immersions are standard. However, in the process of proving Theorem 1, we find a new example of a minimal isometric immersion* $S^3 \longrightarrow S^{15}$ *of degree* 7 *and isotropy order* 2. *(Recall that for degree* 7 *standard immersions of* S^3 *have target dimension* 64.*)*

For the case $n = 3$ we derive the following general existence theorem.

Theorem 2. *For every positive integer d there exists an integer k_d such that for all degrees $k \geq k_d$ there exist $SU(2)$–equivariant minimal isometric immersions of isotropy order d from S^3 into S^k for k even and into S^{2k+1} for k odd.*

Remark. The arguments of [W1, Satz 7.10] extend verbatim in combination with Theorem 2 to show that for $k \geq k_d$ almost all of the known target dimensions of degree k minimal isometric immersions of S^3 into spheres can be realized by immersions of isotropy order d. In particular, for all degrees $k \geq k_d$ there are minimal isometric immersions $S^3 \longrightarrow S^N$ of isotropy order d with target dimension $N = k(k+2) - m_L\, m_R$, where $m_L, m_R \in \{1, 2, \ldots, k+1\}$ for even k and $m_L, m_R \in \{2, 4, \ldots, k+1\}$ for odd k (not both equal to $k+1$).

For the special case of minimal isometric immersions of isotropy order $d = 3$ we conclude $k_3 \leq 42$ from the proof of Theorem 2. We show in a separate argument that $SU(2)$–equivariant immersions of isotropy order $d \geq 3$ are never linearly rigid. Hence we arrive at the following consequences of Theorem 2.

Corollary 2.1. *For all $k \geq k_3$, in particular for all $k \geq 42$, there exist $SU(2)$–equivariant minimal isometric immersions of S^3 into S^k or S^{2k+1} for k even or odd respectively which are not linearly rigid. In particular, for no $k \geq k_3$ is the minimal target dimension of a degree k minimal isometric immersion achieved by an $SU(2)$–equivariant immersion.*

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1. Algebraic and Geometric Preliminaries

Let M be an n–dimensional compact Riemannian manifold and $S^N(r)$ a sphere of dimension N and radius r. A fundamental result of T. Takahashi [T] is the following theorem.

Theorem (Takahashi). *Let M be an n–dimensional compact Riemannian manifold and $F : M \longrightarrow \mathbb{R}^{N+1}$ an isometric immersion. Then F is a minimal isometric immersion into a round sphere if and only if all components of F are eigenfunctions of the Laplace operator on M with respect to the same eigenvalue.*

Therefore the main idea in constructing minimal isometric immersions of a manifold M into a sphere is to find eigenvalues of the Laplacian on M of sufficiently high multiplicity in order to provide the coordinate functions of the immersions.

Another result of Takahashi [T] is that all isotropy irreducible homogeneous Riemannian manifolds, i. e. manifolds $M = G/H$ whose isotropy group H acts irreducibly on the tangent space, do admit such immersions. To see this we consider the eigenspace E_λ to a fixed eigenvalue $\lambda > 0$. On E_λ we have the inner product induced by that of $L^2(M)$ and the group G acts on E_λ by isometries. Let $\{f_1, \ldots, f_{N+1}\}$ be an orthonormal basis of E_λ and let $F = (f_1, \ldots, f_{N+1}) : M \longrightarrow \mathbb{R}^{N+1}$. Then $\sum df_\nu \otimes df_\nu$ on the one hand can be regarded as an inner product on E_λ and on the other hand as a metric on M which is the pull back of the standard metric on $\mathbb{R}^{N+1}$ under F. In the first interpretation it is clear that $\sum df_\nu \otimes df_\nu$ must be invariant under the action of G and hence the metric $\sum df_\nu \otimes df_\nu$ on M must be also. But then $\sum df_\nu \otimes df_\nu$ must be a multiple of the given metric on M as both are invariant under the irreducible action of the isotropy group H. This multiple cannot be zero as the functions f_ν are not constant. Therefore, after multiplying the metric on M by a constant, $F : M \longrightarrow \mathbb{R}^{N+1}$ is an isometric immersion, which by our first stated theorem by Takahashi [T] gives rise to a minimal isometric immersion into a sphere. This immersion is called the *standard minimal isometric immersion* of degree k if λ is the k–th nonzero eigenvalue. We call two immersions *congruent* if they differ by an isometry of the ambient space. Note that a different choice for the orthonormal basis for E_λ gives rise to a congruent immersion.

An example of such a homogeneous Riemannian manifold is the n–dimensional sphere, realized as the homogeneous space $SO(n+1)/SO(n)$. The eigenfunctions of $S^n(1)$ are simply restrictions of harmonic homogeneous polynomials on $\mathbb{R}^{n+1}$ to $S^n(1)$. All harmonic homogeneous polynomials of degree k restrict to eigenfunctions on S^n with the same eigenvalue $\lambda_k = k\,(k+n-1)$ and the dimension of this eigenspace E_{λ_k} is equal to $(2k+n-1)(k+n-2)!/\,(k!(n-1)!)$.

In [DW2] the space of all minimal isometric immersions of $S^n(1)$ into $S^N(r)$ was examined in detail, and it was shown that for $n > 2$ there are many minimal isometric immersions besides the standard ones. If we fix $r = \sqrt{n/\lambda_k}$, or equivalently fix the degree k of the harmonic homogeneous polynomials, then these minimal isometric immersions (up to congruence of the ambient space) are parametrized by a compact convex body in a finite dimensional vector space. The following description of this convex body uses a new interpretation of the do Carmo–Wallach construction given by the second author in his dissertation ([W1], [W2]).

Definition 1.1. *Let $F : M \longrightarrow S_W \subset W$ be a minimal isometric immersion of a Riemannian manifold M into a sphere S_W of radius r of a Euclidean vector space W and consider the associated canonical map $F^* : W^* \longrightarrow E_\lambda$, $\alpha \longmapsto \alpha \circ F$. The natural extension of F^* to second symmetric powers maps the scalar product $<\,,\,>_W \in \mathrm{Sym}^2 W^*$ of the target space to a well defined symmetric bilinear form:*

$$\mathrm{Sym}^2 F^* :\ \mathrm{Sym}^2 W^* \longrightarrow \mathrm{Sym}^2 E_\lambda, \quad <\,,\,>_W \longmapsto G_F$$

We call G_F the **eigenform of F**.

When F is expressed in components $F = (f_1, \ldots, f_{\dim W})$ with respect to an orthonormal basis of W, the eigenform G_F becomes $G_F = \sum f_\nu \otimes f_\nu$. Note that the eigenform G_F of F only depends on the restriction of the metric $<\,,\,>_W$ to the subspace of W spanned by the image of F. If the immersion F is full, i. e. if the image of F spans all of W, then G_F truly depends just on the congruence class of the immersion F. The eigenform of a minimal isometric immersion is always positive semidefinite by construction.

Our goal is to describe the moduli space of all minimal isometric immersions of M into $S^N(r)$ as a subspace of $\mathrm{Sym}^2 E_\lambda$ using the above described eigenforms. In order to do so we explain how to recover a minimal isometric immersion from its eigenform. We first dualize F^* to obtain a map onto W: $(F^*)^* : E_\lambda^* \longrightarrow W$ where $(F^*)^*(\phi)(\alpha) = \phi(\alpha \circ F)$ for $\phi \in E_\lambda^*$ and $\alpha \in W^*$. We then recover the map F by precomposing with the Dirac delta functional:

$$\begin{aligned} M &\xrightarrow{\delta} E_\lambda^* \xrightarrow{(F^*)^*} W \\ p &\longmapsto \delta_p \longmapsto F(p)\,, \end{aligned}$$

where $\delta_p(f) = f(p)$ for $f \in E_\lambda$. In order to turn $(F^*)^*$ into an isometry we have to divide E_λ^* by the kernel of the eigenform G_F as the eigenform is only positive semidefinite by construction. Let $\mathrm{Ker}\, G_F := \{\phi \in E_\lambda^* : G_F(\phi, \cdot) = 0\}$. The eigenform G_F now induces a positive definite scalar product on $E_\lambda^*/\mathrm{Ker}\, G_F$ which turns $(F^*)^*$ into an isometry:

$$M \xrightarrow{\delta} E_\lambda^* \xrightarrow{\pi} E_\lambda^*/\mathrm{Ker}\, G_F \xrightarrow{\cong} W\,.$$

Hence $\pi \circ \delta$ is congruent to the original map F. Given a positive semidefinite symmetric bilinear form $G \in \mathrm{Sym}^2 E_\lambda$ the above construction uniquely defines the map $\pi \circ \delta$. For this map to be a minimal isometric immersion and hence for G to be an eigenform of a minimal isometric immersion the following conditions must be satisfied.

Eigenform conditions. *A positive semidefinite symmetric bilinear form $G \in \mathrm{Sym}^2 E_\lambda$ is the eigenform of a minimal isometric immersion if and only if the following two conditions hold.*

$$\begin{aligned} &(1) \qquad G(\delta_p, \delta_p) = \frac{\dim M}{\lambda} \ \textit{for all } p \in M\,. \\ &(2)\ \ G(X\,\delta_p, Y\,\delta_p) = g(X, Y) \ \textit{ for all } p \in M \textit{ and } X\,,\, Y \in T_pM\,. \end{aligned} \tag{1.1}$$

Here g is the original metric on M and $X\,\delta_p \in E_\lambda^*$ is the functional $f \longmapsto (Xf)(p)$.

Note that the first condition assures that the image of $\pi \circ \delta$ lies in a sphere of radius $\sqrt{\frac{\dim M}{\lambda}}$ whereas the second condition has to be satisfied for $\pi \circ \delta$ to be an isometric immersion. Minimality then follows by Takahashi's theorem. We now describe the moduli space of do Carmo–Wallach using eigenforms.

Moduli space of do Carmo–Wallach. *Let B_λ be the moduli space of all minimal isometric immersions of a Riemannian manifold M with respect to a fixed eigenvalue $\lambda > 0$. Let $B_\lambda^0 \subset \mathrm{Sym}^2 E_\lambda$ be the vector subspace consisting of solutions G_0 of the equations*

$$\begin{aligned} &(1) \quad\ G_0(\delta_p, \delta_p) \quad\ = 0 \ \textit{ for all } p \in M \\ &(2)\ \ G_0(X\,\delta_p, Y\,\delta_p) = 0 \ \textit{ for all } p \in M \ \textit{ and } X\,,\, Y \in T_pM \end{aligned}$$

and let $B_\lambda^A \subset \mathrm{Sym}^2 E_\lambda$ be the affine subspace corresponding to solutions of the eigenform conditions (1.1)*, thus a translate of B_λ^0. The moduli space B_λ is the intersection of B_λ^A with the positive cone $P^+ E_\lambda := \{G \in \mathrm{Sym}^2 E_\lambda : G \geq 0\}$:*

$$B_\lambda \ = \ P^+ E_\lambda \cap B_\lambda^A$$

and is thus possibly empty. Compactness and convexity of B_λ *follow easily from the geometry of intersections of affine subspaces with the positive cone.*

The main advantage of this new description is that it avoids choosing coordinates for representing some fixed standard minimal isometric immersion. Furthermore it does not even rely on the existence of such a standard immersion. The dependence on this choice of coordinates causes various difficulties in the study of the moduli space and can be avoided altogether in using this new description, see [W2] for more details.

One of our main tools are moduli spaces associated to higher fundamental forms. Consider a minimal isometric immersion $F:\, M \longrightarrow S^N \subset \mathbb{R}^{N+1}$. For every point $p \in M$ we can define a filtration of the target vector space

$$\mathbb{R}F(p) \,=\, \mathfrak{F}^0_p \,\subset\, \mathfrak{F}^1_p \,\subset\, \mathfrak{F}^2_p \,\subset\, \cdots \,\subset\, \mathbb{R}^{N+1}$$

by $\mathfrak{F}^b_p := \mathbb{R}F(p) + \mathrm{span}\{\, (X_1 \cdots X_b F)(p) \,|\, X_1, \ldots, X_b \text{ smooth vector fields on } M\}$. By definition the higher fundamental form $\mathrm{I\!I}^b_{F,p},\ b \geq 1,$ is the composition

$$\mathrm{I\!I}^b_{F,p}:\ \mathrm{Sym}^b TM \,\longrightarrow\, \mathfrak{F}^b_p/\mathfrak{F}^{b-1}_p \,\longrightarrow\, \mathbb{R}^{N+1}$$

of the map $\mathrm{Sym}^b TM \longrightarrow \mathfrak{F}^b_p/\mathfrak{F}^{b-1}_p$, $X_1 \cdots X_b \mapsto [X_1 \cdots X_b F(p)] \mod \mathfrak{F}^{b-1}_p$ with the orthogonal projection of the quotient $\mathfrak{F}^b_p/\mathfrak{F}^{b-1}_p$ into $\mathbb{R}^{N+1}$. In this spirit we define $\mathrm{I\!I}^0_{F,p}(1) = F(p)$. The image of $\mathrm{I\!I}^b_{F,p},\ b \geq 0,$ is called the **b–th osculating subspace** of F at p and is denoted by $\mathfrak{O}^b_F$. It is known that for M a sphere and λ the k–th eigenvalue the filtration $\mathfrak{F}^b_p$ becomes stationary $\mathfrak{F}^b_p = \mathbb{R}^{N+1}$ for $b \geq k$, see for example [GT]. In particular we have an orthogonal decomposition of $\mathbb{R}^{N+1}$ into the sum of the first k osculating subspaces depending of course on the chosen point $p \in M$.

In order to compare the higher fundamental forms belonging to different minimal isometric immersions we consider the symmetric bilinear forms

$$<\mathrm{I\!I}^b_{F,p}, \mathrm{I\!I}^b_{F,p}>(\mathfrak{X}, \mathfrak{Y}) \,:=\, <\mathrm{I\!I}^b_{F,p}(\mathfrak{X}),\, \mathrm{I\!I}^b_{F,p}(\mathfrak{Y})>_{\mathbb{R}^{N+1}} \qquad \mathfrak{X}, \mathfrak{Y} \in \mathrm{Sym}^b TM\,.$$

We now specialize to $M = S^3$ and denote the higher fundamental forms of the standard minimal isometric immersion by $\mathrm{I\!I}^b_{\mathrm{std},p}$. The associated bilinear forms $<\mathrm{I\!I}^b_{\mathrm{std},p}, \mathrm{I\!I}^b_{\mathrm{std,p}}>$ can also be defined algebraically and are in this sense independent of the point $p \in S^3$. The following concept due to G. Toth [To3] compares a given minimal isometric immersion with the standard one.

Definition 1.2. *A minimal isometric immersion* $F:\, S^3 \longrightarrow S^N \subset \mathbb{R}^{N+1}$ *is said to have* **isotropy order $\mathbf{d \geq 1}$** *if for all points* $p \in S^3$ *and all* $0 \leq b \leq d$ *the bilinear forms of* F *and those of the standard immersion coincide on* $\mathrm{Sym}^b TS^3$ *:*

$$<\mathrm{I\!I}^b_{F,p},\, \mathrm{I\!I}^b_{F,p}> \,=\, <\mathrm{I\!I}^b_{\mathrm{std}},\, \mathrm{I\!I}^b_{\mathrm{std}}>\ .$$

For $d = 1$ *these equations are clearly equivalent to the eigenform conditions* (1.1).

Various notions of rigidity have been defined for a minimal isometric immersion depending on to the geometry of the moduli space B_λ in a neighborhood of the associated eigenform. The first definitions go back to the work of do Carmo and Wallach [DW1], [DW2].

Definition 1.3. *A minimal isometric immersion* $F : M \longrightarrow S^N$ *is said to be* **(globally) rigid** *if whenever* $\tilde{F} : M \longrightarrow S^N$ *is another minimal isometric immersion then* $\tilde{F} = A \circ F$ *for some* $A \in O(N+1)$ *.*

Remark. Note that immersions which differ by a precomposition with an isometry of the domain may lead to non-congruent immersions. For $M = S^n(1)$, the above notion of rigidity is consistent with the definition of the moduli space B_{λ_k} by do Carmo and Wallach. In this case, rigidity of f amongst degree k minimal isometric immersions implies that the N–stratum within B_{λ_k}, i. e. the set of congruence classes of degree k minimal isometric immersions from $S^n(1)$ to $S^N(r_k)$, collapses to a point.

There is also a weaker notion of rigidity first introduced by N. Wallach [Wa] and later used in [TZ] to analyze the fine structure of the boundary of the convex body.

Definition 1.4. *An isometric immersion* $F : M \longrightarrow S^N \subset \mathbb{R}^{N+1}$ *is said to be* **linearly rigid** *if whenever there exists a square matrix* $A \in M(N+1, \mathbb{R})$ *such that*

(1) *the image* $\mathrm{Im}\,(A \circ F)$ *of* $A \circ F$ *lies in* S^N *and*
(2) $A \circ F : M \longrightarrow \mathbb{R}^{N+1}$ *is an isometric immersion,*

then A *is already an orthogonal matrix* $A \in O(N+1)$ *. Note that for a minimal isometric immersion rigidity implies linear rigidity.*

The following proposition characterizes linear rigidity in terms of the eigenform G_F of a minimal isometric immersion $F : M \longrightarrow S^N$. Recall that the image $\mathrm{Im}\ G_F := \{\, G_F(\phi, \cdot\,) \in E_\lambda : \phi \in E^*_\lambda \} = \mathrm{Im}\ F^*$ is the smallest subspace of E_λ such that $G_F \in \mathrm{Sym}^2 \mathrm{Im}\ G_F \subset \mathrm{Sym}^2 E_\lambda$. Alternatively $\mathrm{Im}\ G_F$ can be defined as the annihilator of the kernel of G_F in E_λ, i. e. $\mathrm{Im}\ G_F := (\mathrm{Ker}\ G_F)^\perp$.

Proposition 1.5. *A minimal isometric immersion* $F : M \longrightarrow S^N \subset \mathbb{R}^{N+1}$ *is linearly rigid if and only if* $\mathrm{Sym}^2 \mathrm{Im}\ G_F$ *and the parameter space* B^0_λ *have trivial intersection, i. e.* $\mathrm{Sym}^2 \mathrm{Im}\ G_F \cap B^0_\lambda = \{0\}$ *.*

Proof. Suppose $\mathrm{Sym}^2 \mathrm{Im}\ G_F \cap B^0_\lambda = \{0\}$ and the composition $A \circ F$ of F with a linear map $A : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}$ is an isometric immersion of M into a sphere. The induced map of $A \circ F$ maps $(\mathbb{R}^{N+1})^*$ into $\mathrm{Im}\,(A \circ F)^* \subset \mathrm{Im}\ F^* \subset E_\lambda$, hence all components of $A \circ F$ are eigenfunctions for the eigenvalue $\lambda > 0$ and $A \circ F$ is minimal by Takahashi's theorem. Its eigenform $G_{A \circ F} \in B_\lambda$ lies in $\mathrm{Sym}^2 \mathrm{Im}\ G_F$ and so the difference $G_F - G_{A \circ F}$ vanishes, because $\mathrm{Sym}^2 \mathrm{Im}\ G_F \cap B^0_\lambda = \{0\}$. Thus F and $A \circ F$ are congruent and $A \in O(N+1)$ by the definition of congruence.

Conversely choose some nontrivial $G^0 \in \mathrm{Sym}^2 \mathrm{Im}\ G_F \cap B^0_\lambda \neq \{0\}$. Since G_F is positive definite on $E^*_\lambda / \mathrm{Ker}\ G_F$ and $G^0 \in \mathrm{Sym}^2 \mathrm{Im}\ G_F$, the symmetric bilinear form $G_F + t\, G^0$ is positive semidefinite for all t sufficiently close to 0. From $\mathrm{Im}\,(G_F + t\, G^0) \subset \mathrm{Im}\ G_F$ or equivalently $\mathrm{Ker}\,(G_F + t\, G^0) \supset \mathrm{Ker}\ G_F$ we conclude that we have a factorization $\tilde{\pi}$ of

$$M \xrightarrow{\ \delta\ } E^*_\lambda \xrightarrow{\ \pi\ } E^*_\lambda / \mathrm{Ker}\ G_F \xrightarrow{\ \tilde{\pi}\ } E^*_\lambda / \mathrm{Ker}\,(G_F + tG^0)\,.$$

By construction $\pi \circ \delta$ is congruent to F, and similarly $\tilde{\pi} \circ \pi \circ \delta$ is congruent to some other minimal isometric immersion $\tilde{F} : M \longrightarrow S^{\tilde{N}} \subset \mathbb{R}^{\tilde{N}+1}$ with a different eigenform $G_F + tG^0 \neq G_F$. The $(N+1)\times(\tilde{N}+1)$–matrix A expressing $\tilde{\pi}$ is thus not orthogonal, nevertheless $A \circ F = \tilde{F}$ is an isometric immersion. $\square$

2. The System of Quadratic Equations

In order to prove Theorems 1 and 2 we first derive a necessary and sufficient condition for an $SU(2)$–equivariant minimal isometric immersion to be of isotropy order $d \geq 1$. This condition can be thought of as a geometric interpretation of the well–known Clebsch–Gordan–formula [FH]. Namely denote by R the standard representation of $SU(2)$. Then the Clebsch–Gordan–formula states that the tensor product of the irreducible complex $SU(2)$–representation $\mathrm{Sym}^k R$ with itself decomposes into the irreducible subspaces

$$\mathrm{Sym}^k R \otimes \mathrm{Sym}^k R \;\cong\; \mathbb{C} \oplus \mathrm{Sym}^2 R \oplus \mathrm{Sym}^4 R \oplus \cdots \oplus \mathrm{Sym}^{2k} R\,. \tag{2.1}$$

Our goal is to give explicit formulas for a set of projections onto these $SU(2)$–invariant subspaces of $\mathrm{Sym}^k R \otimes \mathrm{Sym}^k R$. First recall that the standard representation R of $SU(2)$ carries a unique (up to a constant factor) invariant symplectic form ω as well as an invariant quaternionic structure J satisfying $\omega(Jp,\, J\tilde{p}) = \overline{\omega(p,\, \tilde{p})}$ and $\omega(p,\, Jp) \geq 0$ for all $p,\, \tilde{p} \in R$. The natural extensions of ω and J to $\mathrm{Sym}^k R$ will be denoted by the same symbols. The real part $<\cdot,\,\cdot>:= \mathrm{Re}\,[\,\omega(\cdot,\, J\cdot)]$ of the hermitian form $\omega(\cdot,\, J\cdot)$ is an invariant scalar product on the underlying real vector space of $\mathrm{Sym}^k R$.

The best way to think of $\mathrm{Sym}^k R$ is as the space of homogeneous polynomials of degree k in two variables $p,\, q$, where p and $q := Jp$ with $\omega(p,\, q) = 1$ denote a fixed canonical basis of R. In this notation a suitable basis of $\mathrm{Sym}^k R$ consists of monomials $\psi_\mu := \psi_\mu^{(k)} := \frac{1}{\mu!(k-\mu)!} p^\mu q^{k-\mu}$ with $J\psi_\mu = (-1)^{k-\mu}\psi_{k-\mu}$. Note that $SU(2)$ acts transitively on the unit sphere $\{p \in R : \omega(p,\, Jp) = 1\}$ of R and thus on the canonical basis $p,\, q = Jp$ defined above. Hence the two operators

$$\begin{aligned}
\Lambda\cdot : \mathrm{Sym}^k R \otimes \mathrm{Sym}^{\tilde{k}} R &\longrightarrow \mathrm{Sym}^{k+1} R \otimes \mathrm{Sym}^{\tilde{k}+1} R \\
\omega\lrcorner : \mathrm{Sym}^k R \otimes \mathrm{Sym}^{\tilde{k}} R &\longrightarrow \mathrm{Sym}^{k-1} R \otimes \mathrm{Sym}^{\tilde{k}-1} R
\end{aligned}$$

defined by $\Lambda\cdot := p\cdot \otimes q\cdot - q\cdot \otimes p\cdot$ and $\omega\lrcorner := \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q} - \frac{\partial}{\partial q} \otimes \frac{\partial}{\partial p}$ depending a priori on the canonical basis of R are indeed $SU(2)$–invariant. Together with their commutator $[\Lambda\cdot,\, -\omega\lrcorner] = k + \tilde{k} + 2$ on $\mathrm{Sym}^k R \otimes \mathrm{Sym}^{\tilde{k}} R$ they generate an $\mathfrak{sl}_2\mathbb{R}$–algebra acting on $\mathrm{Sym}\, R \otimes \mathrm{Sym}\, R$ and the decomposition (2.1) is simply the intersection of the subspace $\mathrm{Sym}^k R \otimes \mathrm{Sym}^k R$ with $\mathfrak{sl}_2\mathbb{R}$–isotypical subspaces. Therefore the projections onto the different $SU(2)$–invariant subspaces $\mathrm{Sym}^{2l} R,\ 0 \leq l \leq k$, of $\mathrm{Sym}^k R \otimes \mathrm{Sym}^k R$ according to (2.1) can be chosen to be the compositions

$$\mathbf{pr}_{\,2l} :\ \mathrm{Sym}^k R \otimes \mathrm{Sym}^k R \ \stackrel{(\omega\lrcorner)^{k-l}}{\longrightarrow}\ \mathrm{Sym}^l R \otimes \mathrm{Sym}^l R \ \stackrel{\mathfrak{m}}{\longrightarrow}\ \mathrm{Sym}^{2l} R$$

of $(\omega\lrcorner)^{k-l}$ with the symmetric multiplication $\mathfrak{m}$. Note that the projections $\mathbf{pr}_{\,2l}$ are unique up to a constant factor only. Also the $\mathbf{pr}_{\,2l}$ are not truly projections as linear operators ($\mathbf{pr}^{\,2} = \mathbf{pr}$) unless appropriate inverse embeddings of $\mathrm{Sym}^{2l} R$ in $\mathrm{Sym}^k R \otimes \mathrm{Sym}^k R$ are fixed in advance.

Every unit vector $\psi \in \mathrm{Sym}^k R$ naturally gives rise to a smooth map $F_\psi : SU(2) \longrightarrow S^{2k+1} \subset \mathrm{Sym}^k R$, $\gamma \longmapsto \gamma\psi$ where S^{2k+1} is the unit sphere of the underlying Euclidean vector space $\mathrm{Sym}^k R$. Some of these orbits of $SU(2)$ in $\mathrm{Sym}^k R$ are the images of minimal isometric immersions of S^3. Hence we now concentrate on studying specific orbits of $SU(2)$ in $\mathrm{Sym}^k R$. Whenever F_ψ is a minimal isometric immersion there is an eigenform G_{F_ψ} associated to F_ψ as described in Definition 1.1. We now define an eigenform for any unit vector $\psi \in \mathrm{Sym}^k R$ which is a real quadratic expression in ψ and essentially encodes all geometric properties of F_ψ.

Definition 2.1. *Let ψ be a unit vector, $\omega(\psi, J\psi) = 1$, in the underlying real vector space of the complex irreducible representation $\mathrm{Sym}^k R$ endowed with the scalar product $< \cdot, \cdot > := Re[\omega(\cdot, J\cdot)]$. Depending on whether k is even or odd the eigenform G_ψ associated to ψ is defined to be $\frac{1}{2}\psi \cdot J\psi \in \mathrm{Sym}^2\mathrm{Sym}^k R$ or $\frac{1}{2}\psi \wedge J\psi \in \Lambda^2\mathrm{Sym}^k R$ respectively.*

In [W1] it is shown that the two eigenforms $G_{F_\psi} \in \mathrm{Sym}^2 E_\lambda$ and G_ψ associated to a minimal isometric immersion $F_\psi : S^3 \longrightarrow \mathrm{Sym}^k R$ are related, that is either $\mathrm{Sym}^2\mathrm{Sym}^k R$ or $\Lambda^2\mathrm{Sym}^k R$ can be embedded into $\mathrm{Sym}^2 E_\lambda \otimes_{\mathbb{R}} \mathbb{C}$ such that G_ψ becomes G_{F_ψ}. The rather technical proof of this result is based upon a detailed analysis of the eigenspace E_λ as a representation space of the isometry group $SO(4)$ of S^3. For our purposes the following related proposition will be sufficient.

Proposition 2.2. *The smooth map $F_\psi : SU(2) \longrightarrow S^{2k+1} \subset \mathrm{Sym}^k R$ associated to a unit vector $\psi \in \mathrm{Sym}^k R$ is an $SU(2)$–equivariant minimal isometric immersion of isotropy order $d \geq 1$ if and only if the projection of the associated eigenform G_ψ to $\mathrm{Sym}^4 R \oplus \mathrm{Sym}^8 R \oplus \cdots \oplus \mathrm{Sym}^{4d} R$ vanishes.*

Proof of Proposition 2.2. The tangent bundle of S^3 is naturally trivialized by the left–invariant vector fields $X \in \mathfrak{su}(2)$. Hence we may calculate the filtration $\mathfrak{F}^b_\gamma$ and the higher fundamental forms of the smooth map $F_\psi : S^3 \longrightarrow \mathrm{Sym}^k R$, $\gamma \longmapsto \gamma\psi$ using only these left-invariant vector fields. Left–invariance implies

$$(X_1 \cdots X_b F_\psi)(\gamma) \;=\; \gamma X_1 \cdots X_b \psi$$

for all $X_1, \ldots, X_b \in \mathfrak{su}(2)$, so that the filtration and all higher fundamental forms will be left invariant and we can reduce to calculating them at the identity of $SU(2)$. The higher fundamental forms $\mathrm{II}^b_{F_\psi,\mathrm{Id}} : \mathrm{Sym}^b\mathfrak{su}(2) \longrightarrow \mathfrak{F}^b/\mathfrak{F}^{b-1}$ can be lifted to a map $\mathrm{II}^b_{\mathrm{sym}} : \mathrm{Sym}^b\mathfrak{su}(2) \longrightarrow \mathfrak{F}^b$ by symmetrizing over left–invariant vector fields:

$$\mathrm{II}^b_{\mathrm{sym}}(X_1 \cdot \ldots \cdot X_b) \;=\; (X_1 \cdot \ldots \cdot X_b)\psi \;:=\; \frac{1}{b!}\sum_\sigma X_{\sigma(1)} \cdots X_{\sigma(b)}\psi\,.$$

Recall that the adjoint representation $\mathfrak{su}(2)$ of $SU(2)$ is isomorphic to the real subspace $(\mathrm{Sym}^2 R)^{\mathrm{real}}$ of $\mathrm{Sym}^2 R$. Hermite–reciprocity [FH] thus implies

$$\mathrm{Sym}^b\mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C} \;\cong\; \mathrm{Sym}^2\mathrm{Sym}^b R \;\cong\; \mathrm{Sym}^{2b} R \oplus \mathrm{Sym}^{2b-4} R \oplus \cdots$$

Turning now to the proof of Proposition 2.2 assume first that the eigenform G_ψ lies in the kernel of the projection to $\mathrm{Sym}^4 R \oplus \cdots \oplus \mathrm{Sym}^{4d} R$ or equivalently

$G_\psi \in \mathbb{C} \oplus \mathrm{Sym}^{4d+4}R \oplus \mathrm{Sym}^{4d+8}R \oplus \cdots$. Hence for $b+\tilde{b} \leq 2d$ we obtain that for $\mathfrak{X} \in \mathrm{Sym}^b\mathfrak{su}(2)$ and $\mathfrak{Y} \in \mathrm{Sym}^{\tilde{b}}\mathfrak{su}(2)$ the bilinear forms

$$< \mathfrak{X}\psi, \mathfrak{Y}\psi > = \ \mathrm{Re}\, \omega(\mathfrak{X}\psi, J\mathfrak{Y}\psi) \ = \ \frac{1}{2}\Big(\omega(\mathfrak{X}\psi, \mathfrak{Y}J\psi) + (-1)^k\, \omega(\mathfrak{X}J\psi, \mathfrak{Y}\psi) \Big) \quad (2.2)$$

depending a priori on ψ are in fact induced by G_ψ and depend only on the projection of G_ψ to $\mathbb{C}$. Actually, the remaining representations $\mathrm{Sym}^{4d+4t}R,\ t \in \mathbb{N}$ simply do not occur in $\mathrm{Sym}^b\mathrm{Sym}^2R \otimes \mathrm{Sym}^{\tilde{b}}\mathrm{Sym}^2R$ if $b+\tilde{b} \leq 2d$. Consequently all of these bilinear forms are $SU(2)$–invariant! Hence for all $b \leq d$ the image of $(\mathrm{Sym}^{2b}R)^{\mathrm{real}} \subset \mathrm{Sym}^b\mathfrak{su}(2)$ under the symmetrized map $\mathrm{II}^b_{\mathrm{sym}} : \mathrm{Sym}^b\mathfrak{su}(2) \longrightarrow \mathfrak{F}^b$ is orthogonal to $\mathfrak{F}^{b-1}$ and contained in the b–th osculating subspace of F_ψ.

On the other hand one knows that the maximal dimension of the b–th osculating subspace of a minimal isometric immersion of S^3 is $2b+1$, which coincides with the dimension of the b–th osculating subspace for the standard immersion. Consequently the symmetrized map $\mathrm{II}^b_{\mathrm{sym}}$ induces an isomorphism of $(\mathrm{Sym}^{2b}R)^{\mathrm{real}}$ with the b–th osculating subspace of F_ψ

$$\begin{array}{ccc}
(\mathrm{Sym}^{2b}R)^{\mathrm{real}} & \xrightarrow[\cong]{\mathrm{II}^b_{\mathrm{sym}}} & \mathfrak{O}^b_{F_\psi} \\
\cap\big\downarrow & & \big\downarrow\cap \\
\mathrm{Sym}^b\mathfrak{su}(2) & \xrightarrow{\mathrm{II}^b_{\mathrm{sym}}} & \mathfrak{F}^b
\end{array}$$

as well as the following orthogonal decomposition.

$$\mathfrak{F}^b \otimes_{\mathbb{R}} \mathbb{C} \ = \ \mathbb{C} \oplus \mathrm{Sym}^2R \oplus \mathrm{Sym}^4R \oplus \cdots \oplus \mathrm{Sym}^{2b}R\,.$$

In this sense the higher fundamental forms $\mathrm{II}^b_{F_\psi,\mathrm{Id}} : \mathrm{Sym}^b\mathfrak{su}(2) \longrightarrow \mathfrak{F}^b/\mathfrak{F}^{b-1}$ are just the projections to $(\mathrm{Sym}^{2b}R)^{\mathrm{real}}$. As we can apply the above argument equally well to the standard immersion we obtain the following isomorphisms

$$\mathfrak{O}^b_{\text{std, Id}} \ \xleftarrow{\mathrm{II}^b_{\mathrm{sym}}} \ (\mathrm{Sym}^{2b}R)^{\mathrm{real}} \ \xrightarrow{\mathrm{II}^b_{\mathrm{sym}}} \ \mathfrak{O}_{F_\psi,\mathrm{Id}}$$

The scalar products induced on $(\mathrm{Sym}^{2b}R)^{\mathrm{real}}$ by both of these isomorphisms are $SU(2)$–invariant by construction and hence constant multiples of each other. However, these constants cannot be chosen independently for different b, because the scalar products induced by F_ψ on $(\mathrm{Sym}^{2b}R)^{\mathrm{real}}$, $0 \leq b \leq d,$ only depend on a single constant, namely the projection of G_ψ to $\mathbb{C}$. The scalar products induced by the standard immersion are similarly related. If we make the two scalar products agree on $\mathbb{R} = (\mathrm{Sym}^0R)^{\mathrm{real}}$ for $b=0$, then they will agree on all $(\mathrm{Sym}^{2b}R)^{\mathrm{real}}$. But this is easily achieved by scaling the standard metric on S^3 in such a way that the standard immersion goes into a target sphere of radius $< \psi, \psi >= 1$.

Reversing the argument we may assume by induction that $F_\psi : S^3 \longrightarrow \mathrm{Sym}^k R$, $\gamma \longmapsto \gamma\psi$, is an equivariant minimal isometric immersion of isotropy order $d \geq 1$, and its eigenform G_ψ projects to zero in $\mathrm{Sym}^4 R \oplus \cdots \oplus \mathrm{Sym}^{4d-4} R$. The excess relations proved in [W1] for the spheres and in [W2] for arbitrary manifolds ensure that for a minimal isometric immersion of isotropy order $d-1 \geq 0$ the image of the symmetrized map $\mathrm{II}^d_{\mathrm{sym}} : \mathrm{Sym}^d \mathfrak{su}(2) \longrightarrow \mathfrak{F}^d$ is exactly the orthogonal complement of $\mathbb{R} \oplus (\mathrm{Sym}^2 R)^{\mathrm{real}} \oplus \cdots \oplus (\mathrm{Sym}^{2d-2} R)^{\mathrm{real}} \subset \mathfrak{F}^d$. With F_ψ being a minimal isometric immersion of isotropy order d, the restriction of the scalar product on $\mathfrak{F}^d$ to this orthogonal complement is just the $SU(2)$–invariant scalar product induced by the standard immersion.

On the other hand this scalar product pulled back to $\mathrm{Sym}^d \mathfrak{su}(2)$ is related to the eigenform G_ψ by formula (2.2). It depends linearly on G_ψ and the $SU(2)$–invariant scalar product is already induced by the projection of G_ψ to $\mathbb{C}$. Thus the bilinear form induced on $\mathrm{Sym}^d \mathfrak{su}(2)$ by the projection of G_ψ to $\mathrm{Sym}^{4d} R$ must vanish. However formula (2.2) considered as a linear map in G_ψ and restricted to $\mathrm{Sym}^{4d} R$ describes the injective standard map $\mathrm{Sym}^{4d} R \longrightarrow \mathrm{Sym}^2(\mathrm{Sym}^{2d} R)$. Therefore G_ψ must already project to zero in $\mathrm{Sym}^{4d} R$. $\square$

Remarks 2.3.

(1) Note that Proposition 2.2 gives a simple necessary and sufficient condition for the existence of a minimal isometric immersion of isotropy order $d \geq 1$. For example if ψ is invariant under the icosahedral subgroup of $SU(2)$, then so is its eigenform G_ψ and consequently all projections $\mathbf{pr}_{4b} G_\psi$. As there are no non–trivial polynomials in $\mathrm{Sym}^4 R$ and $\mathrm{Sym}^8 R$ fixed by the icosahedral group [DZ], F_ψ must be a minimal isometric immersion of isotropy order $d = 2$.

(2) When k is even the conjugate linear map J defines a real structure on $\mathrm{Sym}^k R$. In particular, $\mathrm{Sym}^k R$ is reducible as a representation of $SU(2)$ over $\mathbb{R}$ and the orbit of any real polynomial, $\psi = J\psi$, in $\mathrm{Sym}^k R$ spans the invariant subspace fixed by J. More generally, if ψ and $J\psi$ are linearly dependent over $\mathbb{C}$, the orbit of ψ spans a real subspace of dimension $k+1$ and F_ψ essentially becomes a map $F_\psi : S^3 \longrightarrow S^k$ into its unit sphere. This is the reason why we are particularly interested in real polynomials, $\psi = J\psi$, when k is even. Even though there is no true analogue of the reality condition for odd k, we call all polynomials of odd degree **real**.

(3) Assume k is even and $\psi = J\psi \in \mathrm{Sym}^k R$ is a real polynomial. The eigenform G_ψ of ψ is related to the eigenform of the polynomial $p\psi \in \mathrm{Sym}^{k+1} R$

$$G_{p\psi} \;:=\; \frac{1}{2}(p\psi) \wedge J(p\psi) \;=\; \frac{1}{2}(p\psi \otimes q\psi \,-\, q\psi \otimes p\psi) \;=\; 2\,(\Lambda\cdot)\, G_\psi\,.$$

Hence if the projection of G_ψ to $\mathrm{Sym}^4 R \oplus \cdots \oplus \mathrm{Sym}^{4d} R$ vanishes, so does the projection of $G_{p\psi}$. We conclude that if F_ψ is a minimal isometric immersion of isotropy order $d \geq 1$ then so is $F_{p\psi}$.

In order to make the condition of Proposition 2.2 more accessible for calculations we express it in coordinates corresponding to the monomial basis ψ_μ of $\mathrm{Sym}^k R$. We first determine the value of $\mathbf{pr}_{2l}$ on the decomposable tensors $\psi_\mu \otimes \psi_\nu$.

$$\begin{aligned}
&\mathbf{pr}_{2l}(\psi_\mu \otimes \psi_\nu) \\
&= \sum_{\tau=0}^{k-l} (-1)^\tau \binom{k-l}{\tau} \Big(\frac{\partial^{k-l}}{\partial p^{k-l-\tau} \partial q^\tau} \psi_\mu \Big) \Big(\frac{\partial^{k-l}}{\partial p^\tau \partial q^{k-l-\tau}} \psi_\nu \Big) \qquad (2.3) \\
&= \sum_{\tau = 0 \vee (\nu - l) \vee (k-l-\mu)}^{(k-l)\wedge\nu\wedge(k-\mu)} (-1)^\tau \binom{k-l}{\tau} \binom{l-k+\mu+\nu}{\nu-\tau} \binom{l+k-\mu-\nu}{l-\nu+\tau} \psi^{(2l)}_{l-k+\mu+\nu}
\end{aligned}$$

where $\vee$ and $\wedge$ are short hand notations for maximum and minimum respectively. Note that taking these maxima and minima as indicated causes the sum on the right hand side to be void unless $|k-\mu-\nu| \leq l$.

Definition 2.4. *The Clebsch–Gordan–coefficients $C^{k,l}_{\mu,\nu}$ are defined for all quadruples $k \geq l, \mu, \nu \geq 0$ satisfying the inequality $|k-\mu-\nu| \leq l$ by*

$$\mathbf{pr}_{2l}(\psi_\mu \otimes \psi_\nu) \;=\; (-1)^\nu \, C^{k,l}_{\mu,\nu} \, \psi^{(2l)}_{l-k+\mu+\nu}$$

Calculation (2.3) *above shows that they are integers defined alternatively by*

$$C^{k,l}_{\mu,\nu} \;:=\; \sum_{\rho=0}^{l} (-1)^\rho \binom{k-l}{\nu-\rho} \binom{l-k+\mu+\nu}{\rho} \binom{l+k-\mu-\nu}{l-\rho} \qquad (2.4)$$

where summation has been shifted from τ to $\rho := \nu - \tau$ and the additional constraints on $\tau = \nu - \rho$ in (2.3) *merely discard trivially vanishing summands from* (2.4).

The Clebsch–Gordan–coefficients enjoy various symmetries which are not apparent from the definition (2.4) but follow easily from properties of the projections $\mathbf{pr}_{2l}$. The properties we need may be summarized as follows.

$$\begin{aligned}
C^{k,l}_{\mu,\nu} &= (-1)^{k-l+\nu+\mu} \, C^{k,l}_{\nu,\mu} && (2.5) \\
C^{k,l}_{k-\mu,k-\nu} &= (-1)^l \, C^{k,l}_{\mu,\nu} && (2.6) \\
\sum_{\mu=0}^{k} \mu!(k-\mu)! \, C^{k,l}_{\mu,k-\mu} &= (k+1)! \, \delta_{l,0} && (2.7)
\end{aligned}$$

Here $\delta_{l,0}$ denotes the standard Kronecker delta function. The first property (2.5) uses the fact that $\omega\lrcorner$ anticommutes with the flip map $\psi \otimes \tilde{\psi} \longmapsto \tilde{\psi} \otimes \psi$, hence $\mathbf{pr}_{2l}(\psi \otimes \tilde{\psi}) = (-1)^{k-l} \mathbf{pr}_{2l}(\tilde{\psi} \otimes \psi)$. Property (2.6) in turn expresses the "reality" $\mathbf{pr}_{2l}(J\psi \otimes J\tilde{\psi}) = J \, \mathbf{pr}_{2l}(\psi \otimes \tilde{\psi})$ of the projections. The last property (2.7) follows from

$$\frac{1}{k!} (\Lambda \cdot)^k (1 \otimes 1) \;=\; \sum_{\mu=0}^{k} (-1)^{k-\mu} \, \mu! \, (k-\mu)! \, \psi_\mu \otimes \psi_{k-\mu} \; .$$

In fact this equality immediately implies $\sum \mu!(k-\mu)! \, C^{k,l}_{\mu,k-\mu} = 0$ for $l > 0$ and inserting $C^{k,0}_{\mu,\nu} = \binom{k}{\mu} \delta_{k,\mu+\nu}$ settles the case $l = 0$.

As a consequence of Proposition 2.2 we obtain an equivalent description of the isotropy condition using a system of quadratic equations.

Corollary 2.5. *The smooth map $F_\psi : SU(2) \longrightarrow S^{2k+1}$ associated to the unit vector $\psi = \sum_{\mu=0}^{k} z_\mu \psi_\mu$ is a minimal isometric immersion of isotropy order $d \geq 1$ if and only if the coordinates $\{z_\mu\}$ of ψ satisfy the following system of quadratic equations*

$$\sum_{\mu-\nu=m} C^{k,2b}_{\mu,k-\nu} z_\mu \overline{z}_\nu \;=\; k!\,\delta_{b,0} \qquad m = -2b,\,\ldots\,,\,2b,\; b = 0,\,\ldots\,,\,d. \tag{2.8}$$

In the case $d = 1$ equations (2.8) *are exactly the isometry equations originally derived in a different formulation by Mashimo* [Ma1].

Proof of Corollary 2.5. Representing $\psi \otimes J\psi \;=\; \frac{1}{2}\psi \cdot J\psi + \frac{1}{2}\psi \wedge J\psi$ as the sum of its symmetrization and antisymmetrization in $\mathrm{Sym}^k R \otimes \mathrm{Sym}^k R$ and using the symmetry property (2.5) of the projections $\mathbf{pr}_{\,2l}$ we find

$$\mathbf{pr}_{\,4b}(\psi \otimes J\psi) \;=\; \mathbf{pr}_{\,4b}\, G_\psi$$

for all $0 \leq b \leq \lfloor \frac{k}{2} \rfloor$ regardless of the parity of k. If we expand $\psi = \sum_{\mu=0}^{k} z_\mu \psi_\mu$ with respect to the monomial basis ψ_μ we find $J\psi = \sum_{\nu=0}^{k}(-1)^{k-\nu}\overline{z}_\nu \psi_{k-\nu}$ and

$$\mathbf{pr}_{\,4b}\, G_\psi \;=\; \sum_{m=-2b}^{2b} \left(\sum_{\mu-\nu=m} C^{k,2b}_{\mu,k-\nu} z_\mu \overline{z}_\nu \right) \psi^{(4b)}_{m+2b}. \tag{2.9}$$

According to Proposition 2.2 the smooth map F_ψ is a minimal isometric immersion of isotropy order $d \geq 0$ if and only if its associated eigenform G_ψ projects to zero under the projection $\mathrm{Sym}^k R \otimes \mathrm{Sym}^k R \longrightarrow \mathrm{Sym}^4 R \oplus \mathrm{Sym}^8 R \oplus \cdots \oplus \mathrm{Sym}^{4d} R$. Using equation (2.9) this condition is equivalent to the system (2.8) except for an additional equation corresponding to $b = 0$ in (2.8), which reads explicitly $\mathbf{pr}_{\,0}(\psi \otimes J\psi) = k! < \psi, \psi >= k!$ and fixes ψ to be a unit vector. $\square$

3. Proof of Theorem 1

We first establish a connection between the isotropy order and linear rigidity of an equivariant minimal isometric immersion.

Proposition 3.1. *If the smooth map $F_\psi : SU(2) \longrightarrow S^k$ or S^{2k+1} associated to a unit vector $\psi \in \mathrm{Sym}^k R$ is a minimal isometric immersion of isotropy order $d \geq 3$, then it is not linearly rigid in the moduli space B_{λ_k} of all minimal isometric immersions. Furthermore, $\mathrm{Sym}^2 \mathrm{Im}\, G_\psi \cap B^0_{\lambda_k}$ is at least nine dimensional.*

Proof of Proposition 3.1. From the Peter–Weyl formula we know that the matrix coefficients of the irreducible complex representations of $SU(2)$ span an L^2–dense subspace in $C^\infty SU(2) \otimes_{\mathbb{R}} \mathbb{C}$:

$$L^2 SU(2) \;=\; \overline{\bigoplus_{k \geq 0} \mathrm{Sym}^k R \otimes \mathrm{Sym}^k R}$$

where $\eta \otimes \psi \in \mathrm{Sym}^k R \otimes \mathrm{Sym}^k R$ is identified with the associated complex matrix coefficient $(\eta \otimes \psi)(\gamma) := \omega(\,\eta,\, \gamma\psi\,)$, $\gamma \in SU(2)$. The above decomposition of

$L^2SU(2)$ is indeed a complete decomposition into eigenspaces for the scalar Laplacian, in particular the k–th eigenspace E_{λ_k} is the real subspace of $\mathrm{Sym}^kR\otimes\mathrm{Sym}^kR$. To decompose its second symmetric power we need the fundamental isomorphism

$$\begin{aligned}\mathrm{Sym}^2(V\otimes W) &\longrightarrow (\mathrm{Sym}^2V\otimes\mathrm{Sym}^2W)\oplus(\Lambda^2V\otimes\Lambda^2W) \\ (v_1\otimes w_1)\cdot(v_2\otimes w_2) &\longmapsto \frac{1}{2}(v_1\cdot v_2\otimes w_1\cdot w_2)\oplus\frac{1}{2}(v_1\wedge v_2\otimes w_1\wedge w_2)\end{aligned} \tag{3.1}$$

and the Clebsch–Gordan–formula (2.1) to find

$$\mathrm{Sym}^2E_{\lambda_k}\otimes_{\mathbb{R}}\mathbb{C}\;\cong\;\mathrm{Sym}^2(\mathrm{Sym}^kR\otimes\mathrm{Sym}^kR)\;\cong\;\bigoplus_{s,t\leq k,\,s+t\,\mathrm{even}}\mathrm{Sym}^{2s}R\otimes\mathrm{Sym}^{2t}R\,.$$

The argument given by do Carmo–Wallach [DW2] shows that the model vector subspace $B^0_{\lambda_k}\subset\mathrm{Sym}^2E_{\lambda_k}$ for the affine space $B^A_{\lambda_k}\subset\mathrm{Sym}^2E_{\lambda_k}$ of solutions of the eigenform conditions (1.1) satisfies

$$B^0_{\lambda_k}\otimes_{\mathbb{R}}\mathbb{C}\;\supset\;\bigoplus_{s,t\leq k,\,s+t\,\mathrm{even},\,|s-t|\geq 4}\mathrm{Sym}^{2s}R\otimes\mathrm{Sym}^{2t}R\,. \tag{3.2}$$

Equality holds as well but is much more difficult to verify. It was only recently proved by Toth [To2].

According to Proposition 1.5 we need to show $B^0_{\lambda_k}\cap\mathrm{Sym}^2\mathrm{Im}\,F^*_\psi\neq\{0\}$ to prove that F_ψ is not linearly rigid in the moduli space B_{λ_k}. Thus our first objective is to determine the image of the canonical map $F^*_\psi:\mathrm{Sym}^kR\longrightarrow E_{\lambda_k}$. Recall that $F_\psi:SU(2)\longrightarrow\mathrm{Sym}^kR$, $\gamma\mapsto\gamma\,\psi$. In contrast to the complex matrix coefficients considered above, F^*_ψ maps to real matrix coefficients by definition. More precisely we have

$$(F^*_\psi\eta)(\gamma)\;:=\;\mathrm{Re}\,\omega(\eta,\gamma\psi)\;=\;\frac{1}{2}\left(\,\omega(\eta,\gamma\psi)\,+\,\omega(J\eta,\gamma J\psi)\,\right),$$

where $\eta\in\mathrm{Sym}^kR$, $\gamma\in SU(2)$ and ω is the symplectic form on R. Hence $\mathrm{Im}\,F^*_\psi$ is spanned by $(\eta\otimes\psi\,+\,J\eta\otimes J\psi)\in\mathrm{Sym}^kR\otimes\mathrm{Sym}^kR$ and is thus the real subspace of $\mathrm{Sym}^kR\otimes\psi\,+\,\mathrm{Sym}^kR\otimes J\psi$. Note that as expected this real subspace has dimension $k+1$ when ψ and $J\psi$ are linearly dependent over $\mathbb{C}$ and dimension $2k+2$ otherwise. Using the fundamental isomorphism (3.1) above we observe that $\mathrm{Sym}^2\mathrm{Im}\,F^*_\psi$ contains the element

$$\begin{aligned}&\frac{1}{2}(\eta\otimes\psi+J\eta\otimes J\psi)^2\,+\,\frac{1}{2}(i\,\eta\otimes\psi-i\,J\eta\otimes J\psi)^2 \\ &\qquad\qquad=\;(\eta\cdot J\eta\otimes\psi\cdot J\psi)\,\oplus\,(\eta\wedge J\eta\otimes\psi\wedge J\psi)\,.\end{aligned}$$

Replacing η by $J\eta$ we see that $(\eta\cdot J\eta\otimes\psi\cdot J\psi)\oplus(-\eta\wedge J\eta\otimes\psi\wedge J\psi)$ is an element of $\mathrm{Sym}^2\mathrm{Im}\,F^*_\psi$ as well. Eliminating one of the two summands we conclude

$$\mathrm{Sym}^2\mathrm{Im}\,F^*_\psi\otimes_{\mathbb{R}}\mathbb{C}\;\supset\;\mathrm{span}\,\{\,\eta\cdot J\eta\otimes G_\psi\,\}\;=\;\mathrm{Sym}^2(\mathrm{Sym}^kR)\otimes G_\psi$$

when k is even and similarly $\mathrm{Sym}^2\mathrm{Im}\,F_\psi^* \otimes_{\mathbb{R}} \mathbb{C} \supset \Lambda^2(\mathrm{Sym}^k R) \otimes G_\psi$ when k is odd. According to Proposition 2.2 the minimal isometric immersion F_ψ is of isotropy order $d \geq 3$ if and only if its eigenform G_ψ satisfies

$$G_\psi \;\in\; \mathbb{C} \oplus \mathrm{Sym}^{16} R \oplus \mathrm{Sym}^{20} R \oplus \cdots$$

Comparing this with the decomposition (3.2) of $B^0_{\lambda_k} \otimes_{\mathbb{R}} \mathbb{C}$ one immediately checks

$$\mathrm{Sym}^2\mathrm{Im}\,F_\psi^* \otimes_{\mathbb{R}} \mathbb{C} \quad \supset \quad \mathrm{Sym}^8 R \otimes G_\psi \quad \subset \quad B^0_{\lambda_k} \otimes_{\mathbb{R}} \mathbb{C}$$

with $\mathrm{Sym}^8 R \subset \mathrm{Sym}^2(\mathrm{Sym}^k R)$ for even k and $\mathrm{Sym}^8 R \subset \Lambda^2(\mathrm{Sym}^k R)$ for k odd. Thus the intersection $\mathrm{Sym}^2\mathrm{Im}\,F_\psi^* \cap B^0_{\lambda_k}$ contains at least the nine–dimensional real subspace of $\mathrm{Sym}^8 R \otimes G_\psi$ and from Proposition 1.5 we conclude that F_ψ is not linearly rigid. $\square$

Hence in order to prove Theorem 1 we need to produce solutions to the system of quadratic equations (2.8) in the case of $d = 3$ and $k = 36$.

Claim. *For $k = 36$ there exist real polynomials $\psi = J\psi \in \mathrm{Sym}^{36} R$ which satisfy the system of quadratic equations* (2.8) *for $d = 3$. Namely the following real polynomial satisfies system* (2.8) *when normalized to unit length.*

$$\psi_{36}^{\mathrm{example}} \;=\; 2274470\sqrt{2}\,p^{18}q^{18} - 1467168(p^{11}q^{25} - p^{25}q^{11}) - 7\sqrt{26970}(p^2q^{34} + p^{34}q^2)$$

Of course one can simply insert the given polynomial $\psi_{36}^{\mathrm{example}}$ into (2.8) and check that all equations are satisfied. However there are simplifications available which considerably facilitate verification of the above claim. Note that in equations (2.8) the sum is subject to the condition $\mu - \nu = m$ where μ, ν denote the indices (and hence powers) of the different monomials of the given polynomial. The absolute value of m ranges from 0 to $2\,d$ but only the case of $m = 0$ corresponds to equations with non-zero right hand side. Hence if we choose the monomials of the given polynomial $\psi = \sum_{\mu=0}^k z_\mu \psi_\mu$ sufficiently separated, i. e. such that $z_\mu \overline{z}_\nu = 0$ unless $\mu = \nu$ or $|\mu - \nu| \geq 2d + 1$, then all equations of (2.8) with $m \neq 0$ are trivially satisfied. Note that the chosen polynomial $\psi_{36}^{\mathrm{example}}$ meets this condition for $d = 3$. This type of reasoning leads us to restrict ourselves to the following reduced system of equations.

$$\sum_{\mu=0}^{k} C^{k,2b}_{\mu,k-\mu} |z_\mu|^2 \;=\; k!\,\delta_{b,0} \qquad\qquad b = 0, \ldots, d \tag{3.3}$$

The idea of using separated monomials was employed previously by Mashimo [Ma1] to provide examples of solutions of (2.8) for $d = 1$. Verifying that $\psi_{36}^{\mathrm{example}}$ indeed satisfies the reduced equations (3.3), we conclude that it is already a solution to the full system of equations (2.8) as all its monomials are sufficiently separated.

Remark. The reduced system (3.3) is no longer invariant under the natural action of $SU(2)$, hence the set of solutions of (3.3) is no longer a union of $SU(2)$–orbits.

In fact, the set of solutions of the full system of equations (2.8) is the union of all $SU(2)$–orbits which are completely contained in the set of solutions of the reduced system (3.3).

A particularly important difference between the system (2.8) and the subsystem (3.3) is that there are many real solutions for the latter for any $k \geq 2d \geq 0$ whereas it is known that (2.8) has no real solutions in general, e. g. for $k = 4$ and $d = 1$ [Ma1],[Mr]. In fact property (2.7) of the Clebsch–Gordan–coefficients ensures that any tuple $(z_0, \ldots, z_k)$ of complex numbers satisfying $|z_\mu|^2 = \frac{\mu!(k-\mu)!}{k+1}$ is a solution to (3.3). Corresponding to these solutions are real polynomials such as

$$\psi \;=\; \frac{1}{\sqrt{k+1}} \sum_{\mu=0}^{k} \frac{\varepsilon_\mu}{\sqrt{\mu!(k-\mu)!}}\, p^\mu q^{k-\mu} \qquad\qquad \varepsilon_\mu \;=\; \begin{cases} 1 & \mu \text{ even} \\ i & \mu \text{ odd} \end{cases} \tag{3.4}$$

Before we proceed to prove Theorem 2 in the next section, let us explain how we came to choose this particular polynomial $\psi_{36}^{\text{example}}$. In fact, system (3.3) is quite accessible by way of computer experimentation as it is linear in the variables $|z_\mu|^2$. Checking configurations with sufficiently separated monomials we were able to find new examples of real and non–real solutions for low k and d, e. g. the polynomial $\psi_7^{\text{example}} := \frac{\sqrt{42}}{120}\left(\frac{1}{7}p^7 + p^2 q^5\right)$ defining a minimal isometric immersion $S^3 \longrightarrow S^{15}$ of degree $k = 7$ and isotropy order $d = 2$.

Surprisingly, there are no real solutions with sufficiently separated monomials for $d = 3$ in even degrees strictly between $k = 36$ and $k = 42$. Nevertheless in the process of proving Theorem 2 we will see that there do exist real solutions for all even $k \geq 42$. On the other hand a simple argument involving quartic invariants of elements of $\mathrm{Sym}^k R$ gives a lower bound $k \geq 22$ on the degree of any real solution to the system of equations (2.8) with even k. We do not know whether this lower bound is actually realized or not, though the corresponding bounds for $d = 1$ and $d = 2$ are sharp.

4. Proof of Theorem 2

Combinatorial background. For the purpose of analyzing (3.3) we introduce the factorial and binomial polynomials by setting

$$[x]_s \;:=\; \prod_{\nu=0}^{s-1} (x-\nu) = (-1)^s [s-x-1]_s$$

and $\binom{x}{s} := \frac{1}{s!}[x]_s$ for integral $s > 0$ with $\binom{x}{0} := 1$ and $\binom{x}{s} := 0$ for $s < 0$. Note that this definition differs somewhat from the usual extension of the binomial coefficients to real $x \in \mathbb{R}$ using the beta–function. It is more convenient in combinatorial formulas like

$$\sum_{s=0}^{\mu} \binom{s}{t}\binom{x}{\mu-s}\binom{y}{s} = \binom{x+y-t}{\mu-t}\binom{y}{t}, \tag{4.1}$$

where $x,\,y\in\mathbb{R}$ and integral $\mu,t\geq 0$. In fact, using the identity $\binom{s}{t}\binom{y}{s}=\binom{y}{t}\binom{y-t}{s-t}$ (4.1) is immediately reduced to the case of $t=0$, which is a well known combinatorial formula for integral $x,\,y\geq 0$. Using (4.1) it is easy to prove that

$$\begin{aligned}\sum_{s=0}^{\mu}\binom{x+s}{s}\binom{x-y+\mu}{\mu-s}\binom{y-\mu}{s} &= \sum_{s,t=0}^{\mu}\binom{x}{t}\binom{s}{t}\binom{x-y+\mu}{\mu-s}\binom{y-\mu}{s} \\ &= \sum_{t=0}^{\mu}\binom{x}{\mu}\binom{\mu}{\mu-t}\binom{y-\mu}{t} = \binom{x}{\mu}\binom{y}{\mu}\end{aligned}\tag{4.2}$$

using $\binom{x}{t}\binom{x-t}{\mu-t}=\binom{x}{\mu}\binom{\mu}{\mu-t}$ in the second line. Note that the above identities (4.1) and (4.2) are hypergeometric identities and can also be proven using computer certification methods as described in [PWZ].

Lemma 4.1.

$$(-1)^l\,C^{k,l}_{\mu,k-\mu} = \sum_{s=0}^{\mu}(-4)^s\binom{\frac{l}{2}}{s}\binom{\frac{l-1}{2}+s}{s}\binom{k-2s}{\mu-s}\tag{4.3}$$

Proof. Inserting $x=\frac{l-1}{2}$ and $y=l$ into (4.2) and multiplying by $(-1)^\mu[l]_\mu$ we find

$$\begin{aligned}(-1)^\mu[\tfrac{l-1}{2}]_\mu\binom{l}{\mu}^2 &= \sum_{s=0}^{\mu}(-1)^\mu[l]_\mu\binom{\frac{l-1}{2}+s}{s}\binom{-\frac{l+1}{2}+\mu}{\mu-s}\binom{l-\mu}{s} \\ &= \sum_{s=0}^{\mu}\frac{(-1)^s}{s!}\binom{\frac{l-1}{2}+s}{s}[\tfrac{l-1}{2}-s]_{\mu-s}[l]_{2s}\binom{l-2s}{\mu-s} \\ &= \sum_{s=0}^{\mu}\frac{(-1)^s}{s!}\binom{\frac{l-1}{2}+s}{s}[\tfrac{l-1}{2}-s]_{\mu-s}\,4^s\,[\tfrac{l}{2}]_s[\tfrac{l-1}{2}]_s\binom{l-2s}{\mu-s} \\ &= [\tfrac{l-1}{2}]_\mu\sum_{s=0}^{\mu}(-4)^s\binom{\frac{l}{2}}{s}\binom{\frac{l-1}{2}+s}{s}\binom{l-2s}{\mu-s}\end{aligned}\tag{4.4}$$

Now (4.4) is a polynomial identity in l and hence we may divide by $[\frac{l-1}{2}]_\mu$ even if l is odd. From definition (2.4) and property (2.6) we conclude that $(-1)^l\,C^{l,l}_{\mu,l-\mu}=(-1)^\mu\binom{l}{\mu}^2$ and therefore that (4.4) is equivalent to (4.3) in the special case $k=l$. The general case $k\geq l$ then easily follows by induction. □

Before we cast the system of quadratic equations (3.3) into its most useful form, we need to recall the definition and some of the properties of the Bernoulli polynomials $B_b(x)$, $b\geq 0$, and their special values $B_b:=B_b(0)$ at zero, the Bernoulli numbers, compare [Wt]. Apart from index conventions the common definition is with the help of a generating function

$$\frac{te^{tx}}{e^t-1} =: \sum_{b\geq 0}\frac{B_b(x)}{b!}\,t^b \qquad\Longrightarrow\qquad B_b(x) = \sum_{s=0}^{b}\binom{b}{s}B_s\,x^{b-s}$$

Some of the striking properties of the Bernoulli polynomials are summarized in

$$\frac{t}{e^t-1}+\frac{t}{2}=\frac{t}{2}\coth\frac{t}{2} \implies B_1=-\frac{1}{2},\ B_{2b+1}=0,\ b>0 \tag{4.5}$$

$$\frac{te^{tx}}{e^t-1} = -\frac{te^{t(x-1)}}{e^{-t}-1} \implies B_b(x) = (-1)^b\,B_b(1-x) \tag{4.6}$$

$$\frac{te^{t(x+1)}}{e^t-1}-\frac{te^{tx}}{e^t-1}=te^{tx} \implies B_b(x+1)-B_b(x) = bx^{b-1} \tag{4.7}$$

We now define the twisted Bernoulli polynomials used below.

$$\mathfrak{B}_b(x) := \frac{2}{2b+1}\,\frac{B_{2b+1}(\frac{x}{2}+1)}{x+1} \qquad b\geq 0 \tag{4.8}$$

From (4.6) we conclude $B_{2b+1}(\frac{1}{2})=0$ so that $\mathfrak{B}_b(x)$ is indeed a polynomial of degree $2b$ with leading coefficient $\frac{1}{2b+1}\,4^{-b}$, in particular $\mathfrak{B}_0(x)=1$. All twisted Bernoulli polynomials satisfy $\mathfrak{B}_b(x)=\mathfrak{B}_b(-x-2)$ from (4.6) and are divisible by $x(x+2)$ for all $b>0$ due to (4.5). The first few of them read explicitly

$$\mathfrak{B}_0(x) = 1 \qquad \mathfrak{B}_1(x) = \frac{x(x+2)}{12} \qquad \mathfrak{B}_2(x) = \frac{x(x+2)(x^2+2x-\frac{4}{3})}{80}$$

Proposition 4.2. *In terms of the norm square variables* $\zeta_\mu := \frac{1}{\mu!(k-\mu)!}|z_\mu|^2$ *the system of equations* (3.3) *is equivalent to any of the following systems of equations.*

$$\sum_{\mu=0}^{k}\mu!(k-\mu)!\,C^{k,2b}_{\mu,k-\mu}\,\zeta_\mu = k!\delta_{b,0} \qquad b=0,\ldots,d; \tag{3.3}$$

$$\sum_{\mu=0}^{k}\binom{\mu}{b}\binom{k-\mu}{b}\zeta_\mu = \frac{1}{k+1}\binom{k+1}{2b+1} \qquad b=0,\ldots,d; \tag{4.9}$$

$$\sum_{\mu=0}^{k}\left(\frac{k}{2}-\mu\right)^{2b}\zeta_\mu = \mathfrak{B}_b(k) \qquad b=0,\ldots,d. \tag{4.10}$$

Proof. Of course the first system corresponds to (3.3) after changing to the norm square variables $\mu!(k-\mu)!\,\zeta_\mu = |z_\mu|^2$. Next we employ Lemma 4.1 with $l=2b$ to rewrite the coefficients of ζ_μ on the left hand side of (3.3) as

$$\begin{aligned}\mu!\,(k-\mu)!\,C^{k,2b}_{\mu,k-\mu} &= \sum_{s=0}^{b}(-4)^s\binom{b}{s}\binom{b+s-\frac{1}{2}}{s}\binom{k-2s}{\mu-s}\mu!\,(k-\mu)!\\ &= \sum_{s=0}^{b}(-4)^s\,[b]_s\,[b+s-\frac{1}{2}]_s\,(k-2s)!\binom{\mu}{s}\binom{k-\mu}{s}\end{aligned}$$

Note that we need only sum up to $s=b$ in the first line due to the factor $\binom{b}{s}$. As $(-4)^b\,[b]_b\,[2b-\frac{1}{2}]_b\,(k-2b)!$ is never zero, the left hand side of (3.3) is an invertible linear combination of the left hand side of (4.9) and conversely. The right hand side of (4.9) is then fixed by inserting the known solution $\zeta_\mu := \frac{1}{k+1}$ of (3.3) into (4.9). This amounts to checking the combinatorial identity

$$\sum_{\mu=b}^{k-b} \binom{\mu}{b}\binom{k-\mu}{b} \quad = \quad \binom{k+1}{2b+1},$$

for example by comparing dimensions in the canonical isomorphism

$$\bigoplus_{\mu=b}^{k-b} \mathrm{Sym}^{\mu-b}\mathbb{R}^{b+1} \otimes \mathrm{Sym}^{k-\mu-b}\mathbb{R}^{b+1} \cong \mathrm{Sym}^{k-2b}\mathbb{R}^{2b+2}.$$

The proof of the equivalence of the systems (4.9) and (4.10) proceeds along the same lines. In order to show that the left hand sides of (4.9) and (4.10) are again invertible linear combinations of each other we need only observe that

$$\begin{aligned} \binom{\mu}{b}\binom{k-\mu}{b} \quad &= \quad \frac{1}{b!^2}\prod_{s=0}^{b-1}\left(\left(\frac{k}{2}-s\right)^2 - \left(\frac{k}{2}-\mu\right)^2\right) \\ &= \quad \frac{(-1)^b}{b!^2}\left(\frac{k}{2}-\mu\right)^{2b} + \text{ lower order terms in } \left(\frac{k}{2}-\mu\right)^2. \end{aligned}$$

Inserting the known solution $\zeta_\mu := \frac{1}{k+1}$ into (4.10) we find

$$\begin{aligned} \sum_{\mu=0}^{k}(\frac{k}{2}-\mu)^{2b}\zeta_\mu \quad &= \quad \frac{1}{k+1}\sum_{\mu=0}^{k}(\frac{k}{2}-\mu)^{2b} \\ &= \quad \frac{1}{2b+1}\left(\frac{B_{2b+1}(\frac{k}{2}+1)}{k+1} - \frac{B_{2b+1}(-\frac{k}{2})}{k+1}\right) \quad = \quad \mathfrak{B}_b(k) \end{aligned}$$

where we have used (4.7) to convert the sum $\sum(\frac{k}{2}-\mu)^{2b}$ into a telescoping sum. □

Proof of Theorem 2. Using Proposition 2.2 or Corollary 2.5 we can state two equivalent formulations of Theorem 2, an algebraic and a geometric version:

Algebraic form of Theorem 2. *For any isotropy order $d \geq 0$ there exists a $k_d < \infty$ such that real solutions $\psi \in \mathrm{Sym}^k R$ to the system of quadratic equations* (2.8) *exist for all $k \geq k_d$.*

Geometric form of Theorem 2. *For any isotropy order $d \geq 0$ there exists a critical degree $k_d < \infty$ such that $SU(2)$–equivariant minimal isometric immersions $S^3 \longrightarrow S^k$ of isotropy order d exist for all even $k \geq k_d$. Moreover $SU(2)$–equivariant minimal isometric immersions $S^3 \longrightarrow S^{2k+1}$ of isotropy order d exist for arbitrary $k \geq k_d$.*

The special case $d = 1$ of Theorem 2 corresponding to minimal isometric immersions without any isotropy condition was proven by Mashimo [Ma1]. In particular he showed that $k_1 = 5$. For $d = 2$, Theorem 2 may be deduced from the fact that for all even $k \geq 60$ there exist non–trivial polynomials $0 \neq \psi \in \mathrm{Sym}^k R$ fixed by the icosahedral group, which consequently solve the system of equations (2.8), compare Remarks 2.3. We are particularly interested in the case $d = 3$, which we need in order to complete the proof of Corollary 2.1.

We prove Theorem 2 in two steps. First we find a simple sufficient condition for the existence of k_d by using a suitable generalization of the approach of Mashimo [Ma1] in his proof of the special case $d = 1$. In particular we show that under this approach the sequence of systems of equations (4.10) converges for $k \longrightarrow \infty$ towards an asymptotic system (4.11). Given a suitable solution to this asymptotic system one can determine $k_d < \infty$ and write down a general solution to (2.8) for all $k \geq k_d$. This part of the proof is essentially constructive and allows us to write down general solutions for $d = 2$ and $d = 3$ for all even $k \geq 20$ or $k \geq 42$, respectively.

In the second step we prove that the asymptotic system (4.11) can be solved for all $d \geq 0$. However this part of the proof is no longer constructive. Explicit computer calculations to find solutions to the asymptotic system are displayed in Table (4.14).

Proof. Due to Remarks 2.3 it is sufficient to consider only the case of even k. Consequently we restrict to this case although most of the arguments remain valid with only minor modification for odd k. Let us try to find solutions $(\zeta_0, \ldots, \zeta_k)$ to the system of equations (4.10) by setting

$$\zeta_\mu \quad = \quad \begin{cases} \xi_\nu & \text{if } \ \mu = \mu(\nu) := \frac{k}{2} - \nu \lfloor \frac{k}{2m} \rfloor \ \text{ for } \ \nu = 0, \ldots, m \\ 0 & \text{else} \end{cases}$$

where our choice depends on the parameter $m \geq 0$ and $k \geq 2m$ is tacitly understood. Note that the non–vanishing monomials of polynomials ψ corresponding to this choice are indexed by $\mu(\nu)$, $0 \leq \nu \leq m$. Hence these polynomials will have sufficiently separated monomials as soon as $k \geq 2m\,(2d+1)$. In terms of this choice the system of equations (4.10) reads

$$\sum_{\nu=0}^{m} \left(\frac{\nu}{m}\right)^{2b} \xi_\nu \quad = \quad \frac{\mathfrak{B}_b(k)}{m^{2b} \lfloor \frac{k}{2m} \rfloor^{2b}} \quad \xrightarrow{k \to \infty} \quad \frac{1}{2b+1} \qquad b = 0, \ldots, d \qquad (4.11)$$

To check convergence recall that the leading term of the polynomial $\mathfrak{B}_b(k)$ is $\frac{4^{-b}}{2b+1} k^{2b}$ and note that $\lfloor \frac{k}{2m} \rfloor^{2b}$ behaves asymptotically like the polynomial $\left(\frac{k}{2m}\right)^{2b}$. Alternatively consider the finite number of possible residue classes of k modulo $2m$ separately. Thus the crucial property of the system (4.11) is that the right hand sides converge for $k \longrightarrow \infty$ towards asymptotic right hand sides whereas the left hand sides are completely independent of k. The resulting asymptotic system (4.11) is a system of linear equations in ξ_ν with some peculiar properties. A more detailed study of these properties is given in Lemma 4.3 below.

In particular, we show in Lemma 4.3 that for all $d \geq 0$ there exists an $m \geq d$ and a strictly positive solution $(\xi_0, \ldots, \xi_m)$, $\xi_\nu > 0$, of the asymptotic system (4.11). For the special cases $d = 1$, 2 and 3 this statement is easily verified, because the unique solution $(\xi_0, \ldots, \xi_m)$ of the asymptotic system with $m = d$ is already strictly positive. However this is no longer true for $d \geq 4$ and we definitely have to allow m to be greater than d, see (4.14).

Taking Lemma 4.3 for granted we can choose $m \geq d$ and a strictly positive solution $(\xi_0, \ldots, \xi_m)$, $\xi_\nu > 0$, of the asymptotic system (4.11). However all of the maximal $(d+1) \times (d+1)$–minors of the coefficient matrix of system (4.11) are Vandermonde matrices. Consequently the linear map corresponding to this coefficient matrix is surjective, so that we can find non–negative solutions for an entire neighborhood of the vector $(1, \frac{1}{3}, \ldots, \frac{1}{2d+1})$.

With the exact right hand sides of equations (4.11) converging asymptotically towards this vector we can consequently solve the exact system for non–negative $(\xi_0(k), \ldots, \xi_m(k))$ provided $k \gg 0$. For all $k \geq 2m$ for which such a solution is defined, the real polynomial

$$\begin{aligned}\psi(k) := &\frac{\varepsilon_{\mu(0)}}{\mu(0)!}\sqrt{\xi_0(k)}p^{\mu(0)}q^{\mu(0)} + \\ &+ \sum_{\nu=1}^{m} \frac{\varepsilon_{\mu(\nu)}}{\sqrt{\mu(\nu)!\,(k-\mu(\nu)\,)!}}\sqrt{\frac{\xi_\nu(k)}{2}}\Big(p^{\mu(\nu)}q^{k-\mu(\nu)} + (-1)^{\mu(\nu)}p^{k-\mu(\nu)}q^{\mu(\nu)}\Big)\end{aligned}$$

with $\varepsilon_\mu = 1$ or $\varepsilon_\mu = i$ depending on the parity of μ as in (3.4) is a real solution to the reduced system of quadratic equations (3.3). If in addition $k \geq 2m\,(2d+1)$, then the monomials of $\psi(k)$ are sufficiently separated and $\psi(k)$ is a real solution to the full system (2.8) as well. Note that we can always choose the functions $\xi_0(k), \ldots, \xi_m(k)$ to be suitable linear combinations of the "almost rational" functions $\frac{\mathfrak{B}_b(k)}{m^{2b}\lfloor\frac{k}{2m}\rfloor^{2b}}$, $b = 0, \ldots, d$, in k. □

Lemma 4.3. *For all $d \geq 0$ there exists an integer $m \geq d$ and a non–negative solution $(\xi_0, \ldots, \xi_m)$, $\xi_\nu \geq 0$, of the asymptotic system of equations associated to d and m*

$$\sum_{\nu=0}^{m}\Big(\frac{\nu}{m}\Big)^{2b}\xi_\nu \;=\; \frac{1}{2b+1} \qquad b = 0, \ldots, d \tag{4.11}$$

such that at least $d+1$ of the ξ_ν are strictly positive. Being a Vandermonde matrix the corresponding $(d+1)\times(d+1)$–minor of the coefficient matrix is invertible. Hence there even exist strictly positive solutions $(\xi_0, \ldots, \xi_m)$, $\xi_\nu > 0$, for this $m \geq d$.

Proof. The leitmotif of the proof is the identity $\int_0^1 x^{2b}dx = \frac{1}{2b+1}$ revealing the close relationship between equations (4.11) and approximate integration schemes such as

$$\int_0^1 p(x)\,dx \;\approx\; \sum_{\nu=0}^{m} p(\frac{\nu}{m})\,\xi_\nu\,.$$

In fact, this approximation scheme is exact for all even polynomials $p(x)$ of degree at most $2d$ if and only if equations (4.11) are satisfied. In this different context the demand for positive weights $\xi_\nu > 0$ arises from considerations of numerical stability. In the rich tradition of this problem several integration schemes have been devised, e. g. the integration scheme of Gauss [S, pp. 127–136], which relies on properties of the Legendre polynomials $L_b(x) := \frac{1}{b!}\frac{d^b}{dx^b}\big(x^b(1-x)^b\big)$ satisfying

$$\int_0^1 L_a(x)\,L_b(x)\,dx \;=\; \frac{\delta_{a,b}}{2a+1} \tag{4.12}$$

Clever but elementary arguments based on (4.12) alone show that the polynomial $L_{d+1}(x)$ changes sign in exactly $d+1$ different real zeroes $0 < x_0 < \cdots < x_d < 1$ with associated strictly positive weights $\xi_0, \ldots, \xi_d > 0$ such that the approximate integration scheme

$$\int_0^1 p(x)\, dx \quad \approx \quad \sum_{\nu=0}^{d} p(x_\nu)\, \xi_\nu \tag{4.13}$$

is exact for all polynomials p of degree less than or equal to $2d+1$. In particular equation (4.13) for the polynomials $p(x) = x^{2b}$, $b = 0, \ldots, d$, is nothing but (4.11) with $\frac{\nu}{m}$ replaced by x_ν. If the x_ν were all rational numbers we could finish the proof by clearing denominators. Unfortunately however, the zeros of the Legendre polynomials are in general irrational.

Nevertheless equations (4.13) for the polynomials $p(x) = x^{2b}$, $b = 0, \ldots, d$, can be interpreted geometrically as stating that the vector $(1, \frac{1}{3}, \ldots, \frac{1}{2d+1})$ lies in the strict interior of the simplex spanned by the vectors $(1, x_\nu^2, \ldots, x_\nu^{2d})$ in the affine hyperplane $(1, *, \ldots, *) \subset \mathbb{R}^{d+1}$. This is clearly an open condition so that we can perturb the ξ_ν and x_ν slightly to make the latter rational without loosing the property of $\xi_\nu > 0$. Consequently we find solutions to equations (4.11) with exactly $d+1$ of the ξ_ν strictly positive when we choose m to be the common denominator of the $x_\nu \in \mathbb{Q} \cap]0, 1[$. $\square$

Unfortunately Lemma 4.3 only guarantees the existence of solutions but does not provide them. In fact, interesting phenomena occur when looking for actual non–negative solutions, which are displayed in the following table calculated with the help of a computer. It shows the smallest possible $m \geq d$ for fixed isotropy order $d \geq 4$ such that there exists a non–negative solution $(\xi_0, \ldots, \xi_m)$, $\xi \geq 0$, of the asymptotic system of equations (4.11). In fact, for each of these computed cases of d, one can already choose strictly positive solutions for the corresponding smallest value of m. We do not know if this continues to hold for larger values of d. Recall that in order to obtain a solution to the full system (2.8) the degree k must satisfy $k \geq 2m\,(2d+1)$.

d	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
m	5	7	9	11	13	16	19	23	26	30	35	39	44	49	55	60	66	73

(4.14)

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