Classifying Seven Dimensional Manifolds of Fixed Cohomology Type

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Abstract

We study seven dimensional manifolds of fixed cohomology type with integer coefficients: $H^0 \cong H^2 \cong H^5 \cong H^7 \cong \mathbb{Z}$, $H^4 \cong \mathbb{Z}_r$, $H^1 = H^3 = H^6 = 0$, simply called manifolds of type r, where $\mathbb{Z}_r$, is a cyclic group of order r generated by the square of the generator of H^2. Such manifolds include the Eschenburg spaces, the Witten manifolds and the generalized Witten manifolds. Most spaces from these three families admit a Riemannian metric of positive sectional curvature or an Einstein metric of positive Ricci curvature. In 1991 M. Kreck and S. Stolz introduced three invariants to classify manifolds M of type r up to homeomorphism and diffeomorphism. In this article, we show that for spin manifolds of type r we can replace two of the homeomorphism invariants by the first Pontrjagin class and the self-linking number of the manifolds. As the formulas of the two latter invariants are in general much easier to compute, this simplifies the classification of these manifolds up to homeomorphism significantly.

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1. Introduction

Manifolds of type r have played an important role in various different areas of differential geometry. We will describe their role in the areas of positive sectional curvature, positive Ricci curvature and Einstein manifolds. The question of finding new examples of compact simply connected homogeneous and inhomogeneous spaces admitting a Riemannian metric with positive sectional curvature has been interesting to geometers since the 1960s. Besides the compact rank one symmetric spaces which always have positive sectional curvature, there are very few known examples, for details see [5],[26],[2],[12],[28],[9],[10],[4]. Here, we focus on Witten manifolds and Eschenburg spaces. An infinite family of Witten manifolds was found by E. Witten [28] in 1981.

These spaces are homogeneous 7-manifolds admitting Einstein metrics of positive Ricci curvature. There is also a notion of generalized Witten manifolds but it is not known whether they admit Einstein metrics. In 1982 J. H. Eschenburg [9] introduced a new construction of spaces, later called biquotients, which are a generalization of homogeneous spaces. Biquotients are inhomogeneous in general. In [9] Eschenburg also described a generalization of an infinite family of homogeneous spaces admitting a Riemannian metric of positive sectional curvature, the so-called Aloff-Wallach spaces [2]. This new infinite family consists of compact simply connected 7-manifolds, now called Eschenburg spaces, which are biquotients and include a subfamily admitting a Riemannian metric of positive sectional curvature.

The topology of biquotients has been studied extensively. In this article, we will focus on the topology of the Eschenburg spaces and the generalized Witten manifolds. Both families have the same cohomology ring structure:

$$H^0 \cong H^2 \cong H^5 \cong H^7 \cong \mathbb{Z},\ H^4 \cong \mathbb{Z}_r,\ H^1 = H^3 = H^6 = 0,$$

where $\mathbb{Z}_r$ is a cyclic group of order $r \geq 1$, and u^2 is a generator of H^4 if u is a generator of H^2. Manifolds satisfying this condition will be called a *manifolds of type* r. The topology of manifolds of type r is discussed in [13],[14],[1],[17],[18],[19],[20],[11],[7]. In 1988, M. Kreck and S. Stolz [13] introduced new invariants, now called *Kreck-Stolz invariants*, and gave a classification of manifolds of type r up to homeomorphism and diffeomorphism. We state all theorems in the orientation preserving case; for the corresponding theorems in the orientation reversing case the linking form and the Kreck-Stolz invariants change signs.

Classification Theorem I ([14],[15]). *Let M and M' be two smooth manifolds of type r which are both spin or both nonspin. Then M is (orientation preserving) diffeomorphic (homeomorphic) to M' if and only if $s_i(M) = s_i(M') \in \mathbb{Q}/\mathbb{Z}$ ($\overline{s}_i(M) = \overline{s}_i(M') \in \mathbb{Q}/\mathbb{Z}$) for $i = 1, 2, 3$.*

M. Kreck and S. Stolz used their classification theorem to classify all Witten manifolds, see [13], as well as the Aloff-Wallach spaces up to homeomorphism and diffeomorphism, see [14]. In the process, they found the first homeomorphic but not diffeomorphic Einstein manifolds admitting positive sectional curvature. In 1997, L. Astey, E. Micha and G. Pastor [1] used Classification Theorem I to classify a particular subfamily of Eschenburg spaces up to homeomorphism and diffeomorphism. In 1997 and 1998, B. Kruggel [18],[19] obtained various homotopy classifications. We will use the following homotopy classification which can be expressed as follows:

Classification Theorem II ([19]). *Let M and M' be two smooth spin manifolds of type odd r with generators u_M and $u_{M'}$ of $H^2(M;\mathbb{Z})$ and $H^2(M';\mathbb{Z})$, respectively. Let $L({u_M}^2, {u_M}^2)$ denote the self-linking number of M.*

- *If* $\pi_4(M) = \pi_4(M') = 0$*,* M *and* M' *are (orientation preserving) homotopy equivalent if and only if*
$$L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2),\ 2r \cdot s_2(M) = 2r \cdot s_2(M')\,.$$
- *If* $\pi_4(M) \cong \pi_4(M') \cong \mathbb{Z}_2$*,* M *and* M' *are (orientation preserving) homotopy equivalent if and only if*
$$L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2),\ r \cdot s_2(M) = r \cdot s_2(M')\,.$$

In [18], B. Kruggel classified all generalized Aloff-Wallach spaces and generalized Witten manifolds with odd order of the fourth cohomology group up to homotopy. Using his Classification Theorem II, he classified all Eschenburg spaces up to homotopy. In 2005, generalized Witten manifolds were classified by C. Escher [11] up to homeomorphism and diffeomorphism. During the same year, B. Kruggel's paper [20] was published based on an earlier preprint in which a new version of the homeomorphism and diffeomorphism classification of the Eschenburg spaces was stated without proof. A generalization of this theorem is the main purpose of our article. Moreover, B. Kruggel gave a method to compute the Kreck-Stolz invariants s_i for almost all Eschenburg spaces, namely for those Eschenburg spaces satisfying condition (C). Two years later, in 2007 T. Chinburg, C. Escher and W. Ziller [7] used B. Kruggel's construction and a program written in Maple and C code to classify all Eschenburg spaces satisfying condition (C) up to homotopy, homeomorphism, and diffeomorphism.

The main purpose of the present paper is to give a general simplification of the homeomorphism and diffeomorphism classification of most manifolds of type r. This classification theorem is divided into two cases: the spin case and the nonspin case:

Theorem A. *Suppose that* M *and* M' *are smooth spin manifolds of type odd* r *with isomorphic fourth homotopy groups. Let* $u_M \in H^2(M;\mathbb{Z})$ *and* $u_{M'} \in H^2(M';\mathbb{Z})$ *be both generators.*

- M *is (orientation preserving) diffeomorphic to* M' *if and only if*
$$L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2),\ s_1(M) = s_1(M'),\ s_2(M) = s_2(M')\,.$$
- M *is (orientation preserving) homeomorphic to* M' *if and only if*
$$L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2),\ p_1(M) = p_1(M'),\ s_2(M) = s_2(M')\,.$$

Theorem B. *Suppose that* M *and* M' *are smooth nonspin manifolds of type* r*. Let* $u_M \in H^2(M;\mathbb{Z})$ *and* $u_{M'} \in H^2(M';\mathbb{Z})$ *be both generators.*

- M *is (orientation preserving) diffeomorphic to* M' *if and only if*
$$L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2),\ s_1(M) = s_1(M'),\ s_2(M) = s_2(M')\,.$$
- M *is (orientation preserving) homeomorphic to* M' *if and only if*
$$L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2),\ \overline{s}_1(M) = \overline{s}_1(M'),\ s_2(M) = s_2(M')\,.$$

All of the above invariants are briefly described as follows. Firstly, the Kreck-Stolz invariants s_i were defined in [13] via a bounding manifold of a manifold of type r. They are elements in $\mathbb{Q}/\mathbb{Z}$. Also, in [14] a generalized definition of the Kreck-Stolz invariants were given without their explicit formulas. The generalized formulas will be computed in Section 2.2 and can be expressed in terms of a bounding manifold W with boundary $\partial W = M$ of type r.

Secondly, $L({u_M}^2, {u_M}^2)$, where $u_M \in H^2(M;\mathbb{Z})$ is a generator, is the self-linking number of M. It is an element in $\mathbb{Q}/\mathbb{Z}$ and can be computed by using the description of the linking form L defined in [3].

Finally, we show in Section 3 that there exists a relation between the characteristic number z^4 defined on a bounding manifold W and the self-linking number defined on its boundary M. This relation is an important ingredient in the proof of the above classification theorems.

The proof of Theorems A and B will be given in Section 4. Moreover, combining the formulas of the Kreck-Stolz invariants as derived in [20] and [7] with Theorem A yields a complete picture of the classification of the Eschenburg spaces. This is described in Section 5.

This work is based upon the second author's doctoral dissertation at Oregon State University. It is a pleasure to thank Stephan Stolz for helpful information regarding the self-linking number.

2. Definitions of Linking form and Kreck-Stolz invariants

Throughout the article we use $H^i(M)$ and $H_i(M)$ to denote cohomology and homology groups with integer coefficients.

2.1. **Linking form.**

The *linking form* of an oriented n-manifold is a bilinear map on the torsion subgroups of $H^p(M)$ and $H^q(M)$ with values in $\mathbb{Q}/\mathbb{Z}$:

$$\mathrm{Tor}(H^p(M)) \times \mathrm{Tor}(H^q(M)) \longrightarrow \mathbb{Q}/\mathbb{Z},$$

where $p+q = n+1$. The following description of the linking form is based on [3].

Remark 2.1. *In* [3]*, Barden defined the linking form in terms of homology. However, we derive an equivalent definition in terms of cohomology using the Poincaré duality isomorphism.*

Let M be an n-manifold having an orientation $\mu \in H_n(M)$ so that $\mu \frown$ gives the duality isomorphism:

$$\mu \frown : H^k(M) \longrightarrow H_{n-k}(M).$$

Let $a \in H^p(M)$ and $b \in H^q(M)$ be torsion elements where $p+q = n+1$. Then $\mu \frown a$ is an element in $H_{q-1}(M)$. Note that since a is torsion, so is $\mu \frown a$. Let β be the Bockstein homomorphism which is associated to the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{j} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Consider the associated long exact sequence:

$$\cdots \longrightarrow H^{q-1}(M;\mathbb{Q}) \xrightarrow{j^*} H^{q-1}(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} H^q(M;\mathbb{Z}) \xrightarrow{i^*} H^q(M;\mathbb{Q}) \longrightarrow \cdots .$$

Suppose that there exist c_1 and c_2 in $H^{q-1}(M;\mathbb{Q}/\mathbb{Z})$ such that $\beta(c_1) = \beta(c_2) = b$. The existence of c_1 and c_2 follows from the fact that b is a torsion element. Now $c_1 - c_2 = j^*(d)$ for some $d \in H^{q-1}(M;\mathbb{Q})$ by exactness. Then

$$\langle c_1, \mu \frown a\rangle - \langle c_2, \mu \frown a\rangle = \langle j^*(d), \mu \frown a\rangle = j\,\langle d, \mu \frown a\rangle\,.$$

Here $\langle \varphi, \alpha\rangle$ represents evaluation of a cohomology class φ at a homology class α. Since $\mu \frown a$ is a torsion element, the above difference is zero. This implies well-definedness of $\langle \beta^{-1}(b), \mu \frown a\rangle \in \mathbb{Q}/\mathbb{Z}$, which is equal to $\langle a \smile \beta^{-1}(b), \mu\rangle \in \mathbb{Q}/\mathbb{Z}$, by the canonical relation between the cup and cap products. This yields the following definition.

Definition 2.2. *Let M be an n-manifold with an orientation μ, and $a \in H^p(M)$, $b \in H^q(M)$ be torsion elements where $p+q = n+1$. Then the* ***linking number*** *of a with b is*

$$L(a,b) := \left\langle a \smile \beta^{-1}(b), \mu\right\rangle \in \mathbb{Q}/\mathbb{Z},$$

where β is the Bockstein homomorphism associated to a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

The analogue of Lemma D in [3] using the cohomological definition of the linking form can be described as follows:

Proposition 2.3. *With the above notations,*

- *L is a non-singular bilinear form on the torsion subgroups of $H^p(M)$ and $H^q(M)$.*
- *$L(a,b) + (-1)^{(p-1)(q-1)}L(b,a) = 0$ where a and b are torsion elements of $H^p(M)$ and $H^q(M)$, respectively.*

Now consider a smooth manifold M of type r and a generator $u \in H^2(M)$. Since $H^4(M) \cong \mathbb{Z}_r$, u^2 is torsion. By Definition 2.2, there exists the linking number of u^2 with itself:

$$L(u^2,u^2) = \left\langle u^2 \smile \beta^{-1}(u^2), [M]\right\rangle,$$

where β is the Bockstein homomorphism associated to a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

In this situation, $\beta : H^3(M;\mathbb{Q}/\mathbb{Z}) \longrightarrow H^4(M;\mathbb{Z})$. This gives rise to the following definition:

Definition 2.4. *For a smooth manifold M of type r with a generator u of $H^2(M)$,*

$$L(u^2,u^2) = \left\langle u^2 \smile \beta^{-1}(u^2), [M]\right\rangle \in \mathbb{Q}/\mathbb{Z}$$

is called the ***self-linking number*** *of $u^2 \in H^4(M)$.*

2.2. **Kreck-Stolz invariants.** In [13], M. Kreck and S. Stolz defined new invariants $s_i(M),\ i = 1, 2, 3$ for a closed 7-manifold M with $H^4(M;\mathbb{Q}) = 0$ together with a class $u \in H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u$ mod 2 (nonspin case). Here we consider a smooth manifold M of type r and u a generator of $H^2(M)$. In this case, $w_2(M) = u$ mod 2 if M is nonspin and as always trivial if M is spin. Also, $H^4(M;\mathbb{Q}) = 0$ since $H^4(M) \cong \mathbb{Z}_r$. Therefore, a smooth manifold of type r satisfies the Kreck-Stolz conditions, and hence we can use the invariants. The construction of the Kreck-Stolz invariants can be described as follows.

Definition 2.5. *Let M be a closed 7-manifold with a class $u \in H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u$ mod 2 (nonspin case). (M, u) is the **boundary of a pair** (W, z) if W is an 8-manifold with $\partial W = M$, and $z \in H^2(W)$ restricts to u on the boundary such that $w_2(W) = 0$ (spin case) or $w_2(W) = z$ mod 2 (nonspin case).*

M. Kreck and S. Stolz showed the existence of a bounding pair (W, z) of (M, u), see [14] for the proof.

Now let M be a closed 7-manifold with $H^4(M;\mathbb{Q}) = 0$ together with a class $u \in H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u$ mod 2 (nonspin case). Then there exists a bounding pair (W, z). For such a pair (W, z), one defines characteristic numbers $S_i(W, z) \in \mathbb{Q}$ as follows:

$$\begin{aligned} S_1(W,z) &:= \left\langle e^{d/2}\widehat{A}(W), [W,M] \right\rangle, \\ S_2(W,z) &:= \left\langle \mathrm{ch}(\lambda(z)-1)e^{d/2}\widehat{A}(W), [W,M] \right\rangle, \\ S_3(W,z) &:= \left\langle \mathrm{ch}(\lambda^2(z)-1)e^{d/2}\widehat{A}(W), [W,M] \right\rangle, \end{aligned}$$

where

- $d = 0$ in the spin case, $d = z$ in the nonspin case.
- $\lambda(z)$ is the complex line bundle over W with first Chern class $c_1(W) = z$.
- ch is the Chern character, i.e. for a line bundle V, $\mathrm{ch}(V) = \exp(c_1(V))$.
- $\widehat{A}(W)$ is the $\widehat{A}$-polynomial of W.
- $[W, M]$ is the relative fundamental class of a pair (W, M).

Here $\langle \varphi, \alpha \rangle$ represents the evaluation of a cohomology class φ at a homology class α. In particular, $\mathrm{ch}(\lambda(z) - 1)$ and $\mathrm{ch}(\lambda^2(z) - 1)$ represent $\exp(z) - 1$ and $\exp(2z) - 1$, respectively, and the first few terms of the $\widehat{A}$-polynomial are

$$\widehat{A_0} = 1,\ \ \widehat{A_1} = -\frac{p_1}{2^3 \cdot 3},\ \text{ and }\ \widehat{A_2} = \frac{-4p_2 + 7{p_1}^2}{2^7 \cdot 3^2 \cdot 5}\ .$$

Note that the cohomology classes

$$e^{d/2}\widehat{A}(W),\ \ \mathrm{ch}(\lambda(z)-1)e^{d/2}\widehat{A}(W),\ \text{ and }\ \mathrm{ch}(\lambda^2(z)-1)e^{d/2}\widehat{A}(W)$$

are elements in $H^8(W)$ which can be viewed as elements in $H^8(W;\mathbb{Q})$. However, they can not be evaluated on the relative fundamental class $[W, M]$. In order to make sense of the above definition, we need some explanation. First, $\mathrm{ch}(\lambda(z)-1)e^{d/2}\widehat{A}(W)$ and $\mathrm{ch}(\lambda^2(z)-1)e^{d/2}\widehat{A}(W)$ are rational linear combinations of $z^2p_1(W)$ and z^4. Consider the long exact sequence for the pair (W,M) with rational coefficients:

$$\cdots \longrightarrow H^4(W,M;\mathbb{Q}) \xrightarrow{j^*} H^4(W;\mathbb{Q}) \xrightarrow{i^*} H^4(M;\mathbb{Q}) \longrightarrow \cdots .$$

Then $j^*: H^4(W,M;\mathbb{Q}) \longrightarrow H^4(W;\mathbb{Q})$ is surjective since $H^4(M;\mathbb{Q})=0$. Hence, z^2 can be regarded as an element in $H^4(W,M;\mathbb{Q})$ under the pullback $(j^*)^{-1}$. Therefore, the cup products $z^2p_1(W)$ and z^4 should be interpreted as $(j^*)^{-1}(z^2)\smile p_1(W)$ and $(j^*)^{-1}(z^2)\smile z^2$, and hence as elements in $H^8(W,M;\mathbb{Q})$. They can then be evaluated on the relative fundamental class $[W,M]$. Since $e^{d/2}\widehat{A}(W)$ involves the term $p_2(W)$, the same argument can not be used. It is not possible to regard $p_2(W)$ as an element in $H^8(W,M;\mathbb{Q})$. However, we can use the Hirzebruch signature theorem to eliminate $p_2(W)$, and $e^{d/2}\widehat{A}(W)$ eventually is a rational linear combination of ${p_1}^2(W)$, $z^2p_1(W)$, z^4, and sign(W), the signature of W. Similarly, ${p_1}^2(W)$ can be regarded as $(j^*)^{-1}(p_1(W))\smile p_1(W)$.

Now one can write down the explicit formulas for S_i as follows:
Spin Case:

$$\begin{aligned}
S_1(W,z) &= -\frac{1}{2^5\cdot 7}\mathrm{sign}(W) + \frac{1}{2^7\cdot 7}p_1^2,\\
S_2(W,z) &= -\frac{1}{2^4\cdot 3}z^2p_1 + \frac{1}{2^3\cdot 3}z^4,\\
S_3(W,z) &= -\frac{1}{2^2\cdot 3}z^2p_1 + \frac{2}{3}z^4,
\end{aligned}$$

Nonspin Case:

$$\begin{aligned}
S_1(W,z) &= -\frac{1}{2^5\cdot 7}\mathrm{sign}(W) + \frac{1}{2^7\cdot 7}p_1^2 - \frac{1}{2^6\cdot 3}z^2p_1 + \frac{1}{2^7\cdot 3}z^4,\\
S_2(W,z) &= -\frac{1}{2^3\cdot 3}z^2p_1 + \frac{5}{2^3\cdot 3}z^4,\\
S_3(W,z) &= -\frac{1}{2^3}z^2p_1 + \frac{13}{2^3}z^4,
\end{aligned}$$

where

- sign(W) is the signature of W.
- $z^2p_1 := \langle (j^*)^{-1}(z^2)\smile p_1(W), [W,M]\rangle$.
- $z^4 := \langle (j^*)^{-1}(z^2)\smile z^2, [W,M]\rangle$.
- $p_1^2 := \langle (j^*)^{-1}(p_1(W))\smile p_1(W), [W,M]\rangle$.

Here $j^*: H^4(W,M;\mathbb{Q}) \longrightarrow H^4(W;\mathbb{Q})$ is the canonical homomorphism. Note that the signature of W is always an integer. It is defined to be the number of positive diagonal entries minus the number of negative ones of the diagonal symmetric matrix $[\langle a_i\smile a_j,[W]\rangle]$ for some basis $\{a_1,...a_r\}$ for $H^4(W;\mathbb{Q})$, see [23] for details.

To obtain the invariants for the boundary M of W, we need the following proposition.

Proposition 2.6. [14, 15] *Let* $S(W,z) = (S_1(W,z), S_2(W,z), S_3(W,z)) \in \mathbb{Q}^3$. *Then*

$\{S(W,z) \mid W$ *is a closed smooth manifold*, $w_2(W) = 0\} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$,

$\{S(W,z) \mid W$ *is a closed smooth manifold*, $w_2(W) = z \mod 2\} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$,

$\{S(W,z) \mid W$ *is a closed topological manifold*, $w_2(W) = 0\} \cong (\frac{1}{28}\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}$,

$\{S(W,z) \mid W$ *is a closed topological manifold*, $w_2(W) = z \mod 2\} \cong (\frac{1}{28}\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Using this proposition and the fact that

$$S_i(W,z) = S_i(\partial W, z|_{\partial W}) + S_i(W - \partial W, z|_{W-\partial W})$$

one obtains the following:
Smooth case:
$S_i(W,z)$ mod $\mathbb{Z}$ depend only on the boundary of (W,z), and one defines

$$s_i(M,u) := S_i(W,z) \mod \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}.$$

Topological case:
Let

$$\overline{S}_1(W,z) = 28 \cdot S_1(W,z),\ \overline{S}_2(W,z) = S_2(W,z),\ \overline{S}_3(W,z) = S_3(W,z).$$

Then we define

$$\overline{s}_i(M,u) := \overline{S}_i(W,z) \mod \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}.$$

Since u is a element in $H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u$ mod 2 (nonspin case), it turns out that these $s_i(M,u)$ and $\overline{s}_i(M,u)$ do not change when we replace u by $-u$. Therefore, if M is a manifold of type r with a generator $u \in H^2(M)$, one has well-defined invariants $s_i(M)$ and $\overline{s}_i(M) \in \mathbb{Q}/\mathbb{Z}$ for $i = 1,2,3$, called *Kreck-Stolz invariants*. Note that s_i is a generalization of the *Eells-Kuiper invariant* defined in [8].

In some situations we may not be able to find an explicit pair (W,z) with the required properties. This means that we may not always be able to find an explicit bounding spin manifold W if the boundary M is spin. In this case we modify the invariants as follows. Let M be a closed 7-manifold with $H^4(M;\mathbb{Q}) = 0$ together with a class $u \in H^2(M)$ as above, W an 8-manifold with $\partial W = M$, $z, c \in H^2(W)$ such that $z|_M = u, c|_M = 0$, and $w_2(W) = c$ mod 2 (spin case) or $w_2(W) = z + c$ mod 2 (nonspin case). We define characteristic numbers:

$$S_1(W,z,c) := \left\langle e^{(c+d)/2}\widehat{A}(W), [W,M] \right\rangle,$$

$$S_2(W,z,c) := \left\langle \mathrm{ch}(\lambda(z)-1)e^{(c+d)/2}\widehat{A}(W), [W,M] \right\rangle,$$

$$S_3(W,z,c) := \left\langle \mathrm{ch}(\lambda^2(z)-1)e^{(c+d)/2}\widehat{A}(W), [W,M] \right\rangle,$$

where $d = 0$ in the spin case and $d = z$ in the nonspin case. With the same argument, these characteristic numbers for closed manifolds depend only on (M, u), in particular we can define $s_i(M, u) = S_i(W, z, c) \mod \mathbb{Z}$, $\overline{s}_i(M, u) = \overline{S}_i(W, z, c) \mod \mathbb{Z}$, and call them *generalized Kreck-Stolz invariants*.

Remark 2.7. *If M is spin, the bounding manifold could be spin or nonspin. On the other hand, there cannot be any spin structure on W if M is nonspin since any spin structure on a compact manifold with boundary induces a spin structure on its boundary, see for example Proposition 2.15 in* [21].

As before, the below proposition is the result of deriving the characteristic numbers in the above definition in terms of the signature of W and suitable characteristic numbers.

Proposition 2.8. *Let M be a closed 7-manifold with $H^4(M;\mathbb{Q}) = 0$ together with a class $u \in H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u$ mod 2 (nonspin case). Suppose that W is an 8-manifold with $\partial W = M$, $z, c \in H^2(W)$ such that $z|_M = u, c|_M = 0$, and $w_2(W) = c$ mod 2 (spin case) or $w_2(W) = z + c$ mod 2 (nonspin case). The following are the explicit formulas of S_i.*

Spin Case:

$$\begin{aligned}
S_1(W, z, c) &= -\frac{1}{2^5 \cdot 7}\mathit{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2 - \frac{1}{2^6 \cdot 3}c^2 p_1 + \frac{1}{2^7 \cdot 3}c^4, \\
S_2(W, z, c) &= -\frac{1}{2^4 \cdot 3}((zc + z^2)p_1 - (zc^3 + 3z^2c^2 + 4z^3c + 2z^4)), \\
S_3(W, z, c) &= -\frac{1}{2^3 \cdot 3}((zc + 2z^2)p_1 - (zc^3 + 6z^2c^2 + 16z^3c + 16z^4)),
\end{aligned}$$

Nonspin Case:

$$\begin{aligned}
S_1(W, z, c) &= -\frac{1}{2^5 \cdot 7}\mathit{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2 - \frac{1}{2^6 \cdot 3}(c^2 + 2zc + z^2)p_1 \\
&\quad + \frac{1}{2^7 \cdot 3}(c^4 + 4zc^3 + 6z^2c^2 + 4z^3c + z^4), \\
S_2(W, z, c) &= -\frac{1}{2^4 \cdot 3}((zc + 2z^2)p_1 - (zc^3 + 6z^2c^2 + 13z^3c + 10z^4)), \\
S_3(W, z, c) &= -\frac{1}{2^3 \cdot 3}((zc + 3z^2)p_1 - (zc^3 + 9z^2c^2 + 31z^3c + 39z^4)),
\end{aligned}$$

where p_1, z^2, c^2, zc can be regarded as elements in $H^4(W, M; \mathbb{Q})$ under the pullback $(j^*)^{-1}$ and the above classes $p_1^2, c^2p_1, c^4, zcp_1$, etc. are abbreviations for the characteristic numbers

$$\begin{aligned}
p_1^2 &:= \left\langle (j^*)^{-1}(p_1(W)) \smile p_1(W), [W, M] \right\rangle, \\
zcp_1 &:= \left\langle (j^*)^{-1}(zc) \smile p_1(W), [W, M] \right\rangle, \textit{etc.}\ .
\end{aligned}$$

Proof. As the calculations are very similar, we will only show how to obtain the above $S_1(W,z,c)$ for the spin case and $S_3(W,z,c)$ for the nonspin case. First, we can see that

$$e^{(c+d)/2}\widehat{A} = (1+\frac{c}{2}+\frac{c^2}{8}+\frac{c^3}{48}+\frac{c^4}{384}+...)(\widehat{A_0}+\widehat{A_1}+\widehat{A_2}+...),$$

for the spin case, and

$$\begin{aligned}\mathrm{ch}(\lambda^2(z)-1)e^{(c+d)/2}\widehat{A} &= \mathrm{ch}(\lambda^2(z)-1)e^{(z+c)/2}\widehat{A}\\ &= (2z+2z^2+\frac{4z^3}{3}+\frac{2z^4}{3}+...)(1+\frac{z}{2}+\frac{z^2}{8}+\frac{z^3}{48}+...)\\ &\quad (1+\frac{c}{2}+\frac{c^2}{8}+\frac{c^3}{48}+...)(\widehat{A_0}+\widehat{A_1}+...)\\ &= (2z+3z^2+\frac{31z^3}{12}+\frac{39z^4}{24}+...)(1+\frac{c}{2}+\frac{c^2}{8}+\frac{c^3}{48}+...)\\ &\quad (\widehat{A_0}+\widehat{A_1}+...),\end{aligned}$$

for the nonspin case. Hence, by the definitions of $S_1(W,z,c)$ and $S_3(W,z,c)$,

$$\begin{aligned}S_1(W,z,c) &= \frac{c^4}{384}\cdot\widehat{A_0}(W)+\frac{c^2}{8}\cdot\widehat{A_1}(W)+\widehat{A_2}(W)\\ &= \frac{1}{2^7\cdot 3}c^4-\frac{1}{2^6\cdot 3}c^2p_1-\frac{4}{2^7\cdot 3^2\cdot 5}p_2+\frac{7}{2^7\cdot 3^2\cdot 5}p_1^2\\ &= \frac{1}{2^7\cdot 3}c^4-\frac{1}{2^6\cdot 3}c^2p_1-\frac{4}{2^7\cdot 3^2\cdot 5}\cdot\frac{(45\cdot\mathrm{sign}(W)+p_1^2)}{7}+\frac{7}{2^7\cdot 3^2\cdot 5}p_1^2\\ &= -\frac{1}{2^5\cdot 7}\mathrm{sign}(W)+\frac{1}{2^7\cdot 7}p_1^2-\frac{1}{2^6\cdot 3}c^2p_1+\frac{1}{2^7\cdot 3}c^4,\end{aligned}$$

for the spin case, and

$$\begin{aligned}S_3(W,z,c) &= (\frac{zc^3}{24}+\frac{3z^2c^2}{8}+\frac{31z^3c}{24}+\frac{39z^4}{24})\widehat{A_0}(W)+(zc+3z^2)\widehat{A_1}(W)\\ &= (\frac{zc^3}{24}+\frac{3z^2c^2}{8}+\frac{31z^3c}{24}+\frac{39z^4}{24})-\frac{(zc+3z^2)p_1}{24}\\ &= -\frac{1}{2^3\cdot 3}((zc+3z^2)p_1-(zc^3+9z^2c^2+31z^3c+39z^4)),\end{aligned}$$

for the nonspin case. □

3. Linking form, first Pontrjagin class, and characteristic numbers

In this section, we will describe a different approach to the linking number. In particular, we can obtain the linking number between two arbitrary elements in $H^n(M)$ of a $(2n-1)$-manifold M by computing an appropriate characteristic number of a bounding compact oriented $2n$-manifold W. We will show that

$$L(u^2,u^2) = z^4(W) \mod \mathbb{Z},$$

where $L(u^2,u^2)$ is the self-linking number of a manifold M of type r with a bounding pair (W,z). Note that this fact was stated in [13] without proof. Moreover, we will see a relation between the first Pontrjagin class, the linking form, and some characteristic numbers.

Let W be compact oriented $2n$-manifold with boundary M. Let $a,b\in H^n(M;\mathbb{Z})$ be torsion elements such that $a=\bar{a}|_{\partial W}$ and $b=\bar{b}|_{\partial W}$ for some $\bar{a},\bar{b}\in H^n(W;\mathbb{Z})$. Consider the following commutative diagram of two long exact sequences for a pair (W,M) with $\mathbb{Z},\mathbb{Q}$ - coefficients:

$$\begin{tikzcd}
H^n(W,M;\mathbb{Z}) \arrow[r] \arrow[d] & H^n(W;\mathbb{Z}) \arrow[r, "f"] \arrow[d, "i^*"] & H^n(M;\mathbb{Z}) \arrow[d, "g"] \\
H^n(W,M;\mathbb{Q}) \arrow[r, "j^*"] & H^n(W;\mathbb{Q}) \arrow[r, "h"] & H^n(M;\mathbb{Q})
\end{tikzcd}$$

Since a is torsion, $h(i^*(\bar{a}))=g(f(\bar{a}))=g(a)=0$. Similarly, $h(i^*(\bar{b}))=0$. Then there exist $x,y\in H^n(W,M;\mathbb{Q})$ such that $j^*(x)=i^*(\bar{a})$ and $j^*(y)=i^*(\bar{b})$, where $j^*: H^n(W,M;\mathbb{Q})\longrightarrow H^n(W;\mathbb{Q})$ and $i^*: H^n(W;\mathbb{Z})\longrightarrow H^n(W;\mathbb{Q})$. First, we define a relation $l:\mathrm{Tor}(H^n(M;\mathbb{Z}))\times\mathrm{Tor}(H^n(M;\mathbb{Z}))\longrightarrow\mathbb{Q}$ as follows:

Definition 3.1. *With the above notations, let*

$$l:\mathit{Tor}(H^n(M;\mathbb{Z}))\times\mathit{Tor}(H^n(M;\mathbb{Z}))\longrightarrow\mathbb{Q}$$

be the relation defined by $l(a,b)=(-1)^n\langle x\smile y,[W,M]\rangle$, *where* $[W,M]$ *is the relative fundamental class of the pair* (W,M).

Note that

$$l(a,b)=\begin{cases}\langle x\smile y,[W,M]\rangle & \text{if dimension of } a \text{ is even,}\\ -\langle x\smile y,[W,M]\rangle & \text{if dimension of } a \text{ is odd,}\end{cases}$$

and l is not well-defined in general. However, after reducing mod $\mathbb{Z}$, we show that

$$\widetilde{l}(a,b):=l(a,b)\mod\mathbb{Z}\ \in\mathbb{Q}/\mathbb{Z}$$

is independent of the choice of $\bar{a}$, $\bar{b}$, x, and y. That is, $\widetilde{l}(a,b)$ is well defined. In particular, we will see that it is indeed the same as the linking form:

$$L(a,b)=\left\langle a\smile\beta^{-1}(b),[M]\right\rangle\ \in\mathbb{Q}/\mathbb{Z},$$

where $\beta: H^{n-1}(M;\mathbb{Q}/\mathbb{Z})\longrightarrow H^n(M;\mathbb{Z})$ is the Bockstein homomorphism which is associated to the short exact sequence: $0\longrightarrow\mathbb{Z}\longrightarrow\mathbb{Q}\longrightarrow\mathbb{Q}/\mathbb{Z}\longrightarrow 0$.

We notice that the following equation

$$\begin{aligned}\langle x\smile y,[W,M]\rangle &= \left\langle (j^*)^{-1}(i^*(\bar{a}))\smile y,[W,M]\right\rangle\\ &= \langle i^*(\bar{a})\smile y,[W,M]\rangle\\ &= \langle \bar{a}\smile y,[W,M]\rangle\ \in\mathbb{Q}\end{aligned}$$

implies that $l(a,b)$ is independent of x. Similarly, it is independent of y as well. To show independence of the choice of $\bar{a}$ and $\bar{b}$, it is convenient to use the following commutative diagram:

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
H^{n-1}(M;\mathbb{Z}) \arrow[r] \arrow[d] & H^n(W,M;\mathbb{Z}) \arrow[r, "k"] \arrow[d, "p"] & H^n(W;\mathbb{Z}) \arrow[r, "f"] \arrow[d, "i^*"] & H^n(M;\mathbb{Z}) \arrow[d, "g"] \\
H^{n-1}(M;\mathbb{Q}) \arrow[r] \arrow[d] & H^n(W,M;\mathbb{Q}) \arrow[r, "j^*"] \arrow[d, "q"] & H^n(W;\mathbb{Q}) \arrow[r, "h"] \arrow[d] & H^n(M;\mathbb{Q}) \arrow[d] \\
H^{n-1}(M;\mathbb{Q}/\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, "\beta"] & H^n(W,M;\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & H^n(W;\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & H^n(M;\mathbb{Q}/\mathbb{Z}) \arrow[d] \\
H^n(M;\mathbb{Z}) \arrow[r] \arrow[d] & H^{n+1}(W,M;\mathbb{Z}) \arrow[r] \arrow[d] & H^{n+1}(W;\mathbb{Z}) \arrow[r] \arrow[d] & H^{n+1}(M;\mathbb{Z}) \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}
$$

where the horizontal and vertical lines come from the long exact sequences for the pair (W,M) with various coefficients and the ones for various spaces associated to the short exact sequence: $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$, respectively. Suppose that b is the image of $\bar{b}_1$ and $\bar{b}_2$ under the map f. Then $\bar{b}_1 - \bar{b}_2 \in \mathrm{Ker}(f) = \mathrm{Im}(k)$. So, we have

$$y_1 - y_2 = (j^*)^{-1}(i^*(\bar{b}_1)) - (j^*)^{-1}(i^*(\bar{b}_2)) = (j^*)^{-1}(i^*(\bar{b}_1 - \bar{b}_2)) \in \mathrm{Im}(p),$$

and hence $\bar{y}_1 = \bar{y}_2 \ \in H^n(W,M;\mathbb{Q}/\mathbb{Z})$, where $q(y_1) = \bar{y}_1$ and $q(y_2) = \bar{y}_2$. This implies that

$$\langle \bar{a} \smile \bar{y}_1, [W,M] \rangle = \langle \bar{a} \smile \bar{y}_2, [W,M] \rangle \ \in \mathbb{Q}/\mathbb{Z}.$$

Hence, $\widetilde{l}(a,b)$ is independent of the choice of $\bar{b}$. Similarly, it is independent of $\bar{a}$. Therefore, $\widetilde{l}(a,b)$ depends only on a,b. Now it remains to show that

$$\widetilde{l}(a,b) = L(a,b).$$

Here $L(a,b)$ is the linking number of a with b. By the commutative diagram above, the construction of $\bar{y}$ implies that there is an element $\hat{y} \in H^{n-1}(M;\mathbb{Q}/\mathbb{Z})$ that maps to $\bar{y}$ under the boundary map: $\delta : H^{n-1}(M;\mathbb{Q}/\mathbb{Z}) \longrightarrow H^n(W,M;\mathbb{Q}/\mathbb{Z})$, and $\hat{y}$ maps to b under the Bockstein homomorphism: $\beta : H^{n-1}(M;\mathbb{Q}/\mathbb{Z}) \longrightarrow H^n(M;\mathbb{Z})$. That is,

$\delta(\hat{y}) = \bar{y}$ and $\beta(\hat{y}) = b$. By exactness and $a = \bar{a}|_{\partial W}$, $\delta(a)$ is trivial. Using the definition of a coboundary δ and its relation to the cup product, we have

$$\delta(a \smile \hat{y}) = (-1)^n(\bar{a} \smile \delta(\hat{y})).$$

Hence,

$$\begin{aligned}
\widetilde{l}(a,b) &= l(a,b) \mod \mathbb{Z} \\
&= (-1)^n \langle \bar{a} \smile y, [W,M] \rangle \mod \mathbb{Z} \\
&= (-1)^n \langle \bar{a} \smile \bar{y}, [W,M] \rangle \mod \mathbb{Z} \\
&= \langle (-1)^n(\bar{a} \smile \delta(\hat{y})), [W,M] \rangle \mod \mathbb{Z} \\
&= \langle \delta(a \smile \hat{y}), [W,M] \rangle \mod \mathbb{Z} \\
&= \langle a \smile \hat{y}, [M] \rangle \mod \mathbb{Z} \\
&= \left\langle a \smile \beta^{-1}(b), [M] \right\rangle \mod \mathbb{Z} \ \in \mathbb{Q}/\mathbb{Z}.
\end{aligned}$$

This gives rise to the following proposition:

Proposition 3.2. *Let W be compact oriented $2n$-manifold with boundary M and $a, b \in H^n(M;\mathbb{Z})$ torsion elements such that $a = \bar{a}|_{\partial W}$ and $b = \bar{b}|_{\partial W}$ for some $\bar{a}, \bar{b} \in H^n(W;\mathbb{Z})$. Then*

$$L(a,b) = l(a,b) \mod \mathbb{Z} \ \in \mathbb{Q}/\mathbb{Z},$$

where $L(a,b)$ is the linking number of a with b and l is defined as above.

Now let M be a smooth manifold of type r with u a generator of $H^2(M)$ and (W,z) a bounding pair of (M,u) in the sense of Definition 2.5. Using the same argument as above, we observe that

$$z^4 = \left\langle (j^*)^{-1}(z^2) \smile z^2, [W,M] \right\rangle = l(u^2, u^2).$$

Moreover, if the first Pontrjagin class of W, $p_1(W)$, restricts to $p_1(M)$ on the boundary, then we obtain the equations:

$$z^2 p_1 = \left\langle (j^*)^{-1}(z^2) \smile p_1(W), [W,M] \right\rangle = l(u^2, p_1(M)),$$

and

$$p_1^2 = \left\langle (j^*)^{-1}(p_1(W)) \smile p_1(W), [W,M] \right\rangle = l(p_1(M), p_1(M)).$$

Therefore, these equations imply the following corollary:

Corollary 3.3. *For a manifold M of type r with a generator u of $H^2(M)$ and a bounding pair (W,z) of (M,u), the self-linking number is*

$$L(u^2, u^2) = z^4 \mod \mathbb{Z}.$$

In particular, if $p_1(W)$ restricts to $p_1(M)$ on the boundary, the following linking numbers hold:

$$L(u^2, p_1(M)) = z^2 p_1 \mod \mathbb{Z},$$

and

$$L(p_1(M), p_1(M)) = p_1^2 \mod \mathbb{Z}.$$

4. HOMEOMORPHISM AND DIFFEOMORPHISM CLASSIFICATION

The classification of manifolds of type r up to homeomorphism and diffeomorphism was originally provided by M. Kreck and S. Stolz [13] in 1988. In 1997 and 1998, B. Kruggel [18],[19] obtained various homotopy classifications for particular subfamilies of manifolds of type r. In this section, we will prove the main result of the present paper. This result is divided into two cases: the spin case and the nonspin case.

4.1. **Spin Case.** Let M be a smooth spin manifold of type r, u a generator of $H^2(M)$, and (W, z) a bounding pair. The following are the Kreck-Stolz invariants for the spin case:

$$\begin{aligned} s_1(M) &= -\frac{1}{2^5 \cdot 7}\mathrm{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2 \mod \mathbb{Z}, \\ s_2(M) &= -\frac{1}{2^4 \cdot 3}z^2 p_1 + \frac{1}{2^3 \cdot 3}z^4 \mod \mathbb{Z}, \\ s_3(M) &= -\frac{1}{2^2 \cdot 3}z^2 p_1 + \frac{2}{3}z^4 \mod \mathbb{Z}. \end{aligned}$$

In order to obtain a new version of the classification theorem, we need appropriate bounding manifolds W and W' of M and M' so that we can compare the terms z^2p_1, z^4, p_1^2, and the signature. Fortunately, the construction of the bounding manifolds in Theorem 2.1 [19] perfectly works in this situation. But we have to start with two spin manifolds that are homotopy equivalent. Hence a special version of our classification theorem can be stated as follows:

Theorem 4.1. *Let M and M' be two smooth spin manifolds of type r. Then*

- *M is (orientation preserving) diffeomorphic to M' if and only if M is (orientation preserving) homotopy equivalent to M' and $s_1(M) = s_1(M')$, $s_2(M) = s_2(M')$.*
- *M is (orientation preserving) homeomorphic to M' if and only if M is (orientation preserving) homotopy equivalent to M' and $p_1(M) = p_1(M')$, $s_2(M) = s_2(M')$.*

Using Classification Theorem II as described in Section 1 for spin manifolds of type odd r, one can replace the homotopy equivalence statement by the self-linking number and s_2. Hence, combining this fact with Classification Theorem I in Section 1 gives the desired classification of Theorem A. Note that this proof only works for smooth spin manifolds of odd type.

Before proving Theorem 4.1 we will describe the construction of bounding manifolds of two (orientation preserving) homotopy equivalent spin manifolds as given in [19], and state some general results.

Let M and M' be two smooth spin manifolds of type r. Suppose they are (orientation preserving) homotopy equivalent, let $h : M' \longrightarrow M$ be a homotopy equivalence and u_M a generator of $H^2(M)$. The construction is done in the PL category. Note that every smooth manifold admits a PL structure by Whitehead's theorem on triangulations [27].

By the existence of a bounding pair, there exists a pair (W, z_w) such that W is a PL 8-manifold with a spin structure, M is its boundary, and $z_w \in H^2(W)$ restricts to u_M on the boundary. Let BPL be the classifying space for piecewise linear vector bundles and $BPL\langle 4\rangle$ its 3-connected cover. Then $BPL\langle 4\rangle$ classifies PL bundles with a spin structure and is equivalent to $BSpin$ in low dimensions. By surgery theory [6] and [22], we can assume that the induced map $W \longrightarrow BS^1 \times BSpin$ given by z_w and the spin structure is a 4-connected. Define

$$Q := W \cup_h M' \times I,$$

where $(x, 0) \in M' \times \{0\}$ is identified with $h(x) \in M \subset W$ and M' is considered as $M' \times \{1\}$, a subspace of Q. Section 2.4 in [16] shows that (Q, M') is a Poincaré pair. By construction, W and Q are homotopy equivalent. Let $\nu : Q \longrightarrow BSG$ be the Spivak normal bundle where BSG is the classifying space of oriented spherical fibrations. Consider M' as a PL manifold. The restriction of ν has a lift to $BSPL$, the classifying space of oriented PL bundles. Then there is a commutative diagram:

$$\begin{tikzcd} M' \arrow[r, "\nu_{M'}"] \arrow[d] & BSPL \arrow[d] \\ Q \arrow[r, "\nu"] & BSG \end{tikzcd}$$

The existence of a PL structure on Q depends on obstructions which lie in the cohomology groups:

$$H^{i+1}(Q, M'; \pi_i(G/PL)) \cong H^{i+1}(Q, M') \otimes \pi_i(G/PL)$$

since $H^*(Q, M')$ is free abelian. One knows that $\pi_i(G/PL)$ is trivial for odd integers i. Also, since $W \longrightarrow BS^1 \times BSpin$ is a 4-connected, $\pi_i(W) = 0$ for $i = 1, 2, 3$. Using the Hurewicz theorem and the universal coefficient theorem, $H^i(W) = H_i(W) = 0$ for $i = 1, 2, 3$. By Lefschetz duality, $H^{i+1}(Q, M') = H^{i+1}(W, M') = H_{8-i-1}(W) = 0$ for even integers i. Therefore, all obstructions vanish. This implies that there exists a lift: $\nu_Q : Q \longrightarrow BSPL$ of ν relative to M'. By the usual transversality arguments and the process of surgery [22], there is a 4-connected degree one normal map:

$$f : (W', M') \longrightarrow (Q, M')$$

relative to M' and the difference of the signatures of W' and Q is divisible by 8, where W' is a bounding PL 8-manifold of M' with the canonical spin structure. Let $z_Q \in H^2(Q)$ be a generator. Then $H^2(W') \cong H^2(Q) \cong \mathbb{Z}$ has a generator $z_{W'} = f^*(z_Q)$ with $z_{W'}|_{M'} = u_{M'}$, a generator of $H^2(M')$. Define the first Pontrjagin class of Q as $p_1(Q) := p_1(-\nu_Q)$. Hence, we have $p_1(W') = f^*(p_1(Q))$. Now consider the long exact sequences for the pairs (W, M), (Q, M'), and (W', M'):

$$\begin{tikzcd}
\cdots \arrow[r] & H^4(W,M) \arrow[r, "j^*"] \arrow[d, "\cong"] & H^4(W) \arrow[r, "i^*"] \arrow[d, "\cong"] & H^4(M) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H^4(Q,M') \arrow[r, "j^*"] \arrow[d, "f^*"] & H^4(Q) \arrow[r, "i^*"] \arrow[d, "f^*"] & H^4(M') \arrow[r] \arrow[d, "f^*"] & \cdots \\
\cdots \arrow[r] & H^4(W',M') \arrow[r, "j^*"] & H^4(W') \arrow[r, "i^*"] & H^4(M') \arrow[r] & \cdots
\end{tikzcd}$$

There exist elements $v_W \in H^4(W,M)$, $v_Q \in H^4(Q,M')$, and $v_{W'} \in H^4(W',M')$ such that

$$j^*(v_W) = r \cdot {z_W}^2,\ j^*(v_Q) = r \cdot {z_Q}^2, \text{and } j^*(v_{W'}) = r \cdot {z_{W'}}^2$$

since these image elements are trivial in $H^4(M) \cong H^4(M') \cong \mathbb{Z}_r$. Similarly, there exist elements $u_W \in H^4(W,M)$, $u_Q \in H^4(Q,M')$, and $u_{W'} \in H^4(W',M')$ such that

$$j^*(u_W) = r \cdot p_1(W),\ j^*(u_Q) = r \cdot p_1(Q), \text{and } j^*(u_{W'}) = r \cdot p_1(W').$$

This construction yields the following lemma:

Lemma 4.2. *Suppose that M and M' are (orientation preserving) homotopy equivalent smooth spin manifolds of type r. With the above notations, the following hold:*

$$r \cdot {z_W}^2 p_1(W) = r \cdot {z_{W'}}^2 p_1(W') \mod 24 \quad \textit{and} \quad r \cdot {z_W}^4 = r \cdot {z_{W'}}^4 \ \in \mathbb{Z}.$$

Moreover, if $p_1(M) = p_1(M') \in \mathbb{Z}_r$, then $r \cdot {p_1}^2(W) = r \cdot {p_1}^2(W') \ \in \mathbb{Z}$.

Proof. By the construction of Q, the restriction of the PL bundle, induced by ν_Q over Q, to W is fiber homotopy equivalent to the PL bundle induced by the classifying map $\nu_W : W \longrightarrow BSPL$ over W. It follows by [24] that $p_1(W) = p_1(Q) \in H^4(W; \mathbb{Z}_{24})$, after the canonical identification of $H^4(W)$ and $H^4(Q)$. This also implies that $v_W p_1(W) = v_Q p_1(Q) \mod 24$, and $v_W {z_W}^2 = v_Q {z_Q}^2 \ \in \mathbb{Z}$, since v_W and v_Q are identical. The properties of a degree one normal map f show that $v_Q p_1(Q) = v_{W'} p_1(W') \ \in \mathbb{Z}$, and $v_Q {z_Q}^2 = v_{W'} {z_{W'}}^2 \ \in \mathbb{Z}$. Hence,

$$\begin{aligned}
r \cdot {z_W}^2 p_1(W) &= \left\langle (j^*)^{-1}(r \cdot {z_W}^2) \smile p_1(W), [W, M] \right\rangle \\
&= v_W p_1(W) = v_Q p_1(Q) \mod 24 = v_{W'} p_1(W') \mod 24 \\
&= \left\langle (j^*)^{-1}(r \cdot {z_{W'}}^2) \smile p_1(W'), [W', M'] \right\rangle \mod 24 \\
&= r \cdot {z_{W'}}^2 p_1(W') \mod 24,
\end{aligned}$$

and

$$\begin{aligned}
r \cdot {z_W}^4 &= \left\langle (j^*)^{-1}(r \cdot {z_W}^2) \smile {z_W}^2, [W, M] \right\rangle \\
&= v_W {z_W}^2 = v_Q {z_Q}^2 = v_{W'} {z_{W'}}^2 \\
&= \left\langle (j^*)^{-1}(r \cdot {z_{W'}}^2) \smile {z_{W'}}^2, [W', M'] \right\rangle \\
&= r \cdot {z_{W'}}^4 \ \in \mathbb{Z}.
\end{aligned}$$

Now we will show the last statement. The definition of $p_1(Q)$ implies that

$$p_1(Q)|_{M'} = i^*(p_1(Q)) = p_1(-\nu_Q|_{M'}) = p_1(-\nu_{M'}) = p_1(M').$$

This is equivalent to say that the first Pontrjagin class $p_1(Q)$ restricts to $p_1(M')$ on M'. Since (W, M) and (Q, M') are homotopy equivalent, $p_1(W)$ restricts to $p_1(M)$ on the boundary M as well. With the same argument as above, if $p_1(M) = p_1(M')$ under the identification $H^4(M) \cong H^4(M')$, then

$$\begin{aligned} r \cdot {p_1}^2(W) &= \big\langle (j^*)^{-1}(r \cdot p_1(W)) \smile p_1(W), [W, M] \big\rangle \\ &= u_W p_1(W) = u_Q p_1(Q) = u_{W'} p_1(W') \\ &= \big\langle (j^*)^{-1}(r \cdot p_1(W')) \smile p_1(W'), [W', M'] \big\rangle \\ &= r \cdot {p_1}^2(W') \ \in \mathbb{Z}. \end{aligned}$$

□

Applying this lemma introduces homotopy invariants $2r \cdot s_2(M)$ and $r \cdot s_3(M)$ for a smooth spin manifold of type r as in the following corollary. Note that the first invariant was proved by B. Kruggel [19], and we can use the similar argument for the second one.

Corollary 4.3. *If M is a smooth spin manifold of the type r, then $2r \cdot s_2(M)$ and $r \cdot s_3(M)$ are (oriented) homotopy invariants.*

Proof. Let M and M' be (orientation preserving) homotopy equivalent smooth spin manifolds of the same type. By the above construction, we have (M, u_M) and $(M', u_{M'})$ are the boundary of the pairs (W, z_W) and $(W', z_{W'})$ such that

$$r \cdot {z_W}^2 p_1(W) = r \cdot {z_{W'}}^2 p_1(W') \mod 24 \quad \text{and} \quad r \cdot {z_W}^4 = r \cdot {z_{W'}}^4 \ \in \mathbb{Z}.$$

These two equations imply that

$$\begin{aligned} 2r \cdot s_2(M) &= 2r \cdot S_2(W, z_W) \\ &= -\frac{r}{2^3 \cdot 3} \cdot {z_W}^2 p_1(W) + \frac{r}{2^2 \cdot 3} \cdot {z_W}^4 \\ &= -\frac{r}{2^3 \cdot 3} \cdot {z_{W'}}^2 p_1(W') + \frac{r}{2^2 \cdot 3} \cdot {z_{W'}}^4 \\ &= 2r \cdot S_2(W', z_{W'}) \\ &= 2r \cdot s_2(M') \ \in \mathbb{Q}/\mathbb{Z}, \end{aligned}$$

and similarly

$$\begin{aligned} r \cdot s_3(M) &= -\frac{r}{2^2 \cdot 3} \cdot {z_W}^2 p_1(W) + \frac{2r}{3} \cdot {z_W}^4 \\ &= -\frac{r}{2^2 \cdot 3} \cdot {z_{W'}}^2 p_1(W') + \frac{2r}{3} \cdot {z_{W'}}^4 \\ &= r \cdot s_3(M') \ \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

□

For any smooth spin manifold of type r, combining the above lemma and Corollary 3.3 determines its self-linking number as follows:

Corollary 4.4. *Suppose that M is a smooth spin manifold of type r and u_M is a generator of $H^2(M)$. Then the self-linking number of M can be written as*

$$L({u_M}^2, {u_M}^2) = \frac{N}{r} \ \in \mathbb{Q}/\mathbb{Z},$$

where N is some integer.

Proof. With the previous manifold W, we have $r \cdot {z_W}^4 \ \in \mathbb{Z}$, and so ${z_W}^4$ is some integer over r. By Corollary 3.3, $L({u_M}^2, {u_M}^2) = {z_W}^4 \mod \mathbb{Z}$. We are done. □

Now we are ready to prove Theorem 4.1 which gives rise to one of the main classifications in this article.

($\Longrightarrow$) This follows from the homeomorphism and diffeomorphism classification, Classification Theorem I, and the fact that the first Pontrjagin class is a homeomorphism invariant, see [25].

($\Longleftarrow$) Using the above notations, the pairs (W, z_W) and $(W', z_{W'})$ are bounding manifolds of (M, u_M) and $(M', u_{M'})$. First, suppose that M and M' are (orientation preserving) homotopy equivalent and $s_2(M) = s_2(M') \in \mathbb{Q}/\mathbb{Z}$. By Lemma 4.2, we have the fact that $r \cdot {z_W}^4 = r \cdot {z_{W'}}^4 \ \in \mathbb{Z}$ which is equivalent to the equation: ${z_W}^4 = {z_{W'}}^4 \ \in \mathbb{Q}$. By the assumption that $s_2(M) = s_2(M') \in \mathbb{Q}/\mathbb{Z}$, we have

$$-\frac{1}{2^4 \cdot 3}{z_W}^2 p_1(W) + \frac{1}{2^3 \cdot 3}{z_W}^4 = -\frac{1}{2^4 \cdot 3}{z_{W'}}^2 p_1(W') + \frac{1}{2^3 \cdot 3}{z_{W'}}^4$$

as elements in $\mathbb{Q}/\mathbb{Z}$. This implies that

$${z_W}^2 p_1(W) = {z_{W'}}^2 p_1(W') \mod (2^4 \cdot 3).$$

Therefore,

$$\begin{aligned} s_3(M) &= -\frac{1}{2^2 \cdot 3}{z_W}^2 p_1(W) + \frac{2}{3}{z_W}^4 \\ &= -\frac{1}{2^2 \cdot 3}{z_{W'}}^2 p_1(W') + \frac{2}{3}{z_{W'}}^4 \\ &= s_3(M') \ \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

Now the condition $s_1(M) = s_1(M') \in \mathbb{Q}/\mathbb{Z}$ gives the complete proof for the diffeomorphism case. For the homeomorphism case, we need to assume further that $p_1(M) = p_1(M') \in \mathbb{Z}_r$. Using Lemma 4.2, it follows that

$$r \cdot {p_1}^2(W) = r \cdot {p_1}^2(W') \ \in \mathbb{Z}$$

which is equivalent to the equation:

$${p_1}^2(W) = {p_1}^2(W') \ \in \mathbb{Q}.$$

By the process of surgery [22], the construction of the normal map f and the PL manifold W' yields that

$$\text{sign}(W) = \text{sign}(Q) = \text{sign}(W') \mod 8,$$

where the first equation holds because W and Q are homotopy equivalent. Therefore,

$$\begin{aligned} 28 \cdot s_1(M) &= -\frac{1}{2^3}\text{sign}(W) + \frac{1}{2^5}{p_1}^2(W) \\ &= -\frac{1}{2^3}\text{sign}(W') + \frac{1}{2^5}{p_1}^2(W') \\ &= 28 \cdot s_1(M') \ \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

Thus, we finish the proof of the homeomorphism case.

4.2. **Nonspin Case.** Let M be a nonspin manifold of type r together with a generator $u \in H^2(M)$. The existence of a bounding pair ensures that there exists a nonspin bounding pair (W, z). The Kreck-Stolz invariants can be described as follows:

$$\begin{aligned} S_1(W,z) &= -\frac{1}{2^5 \cdot 7}\text{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2 - \frac{1}{2^6 \cdot 3}z^2 p_1 + \frac{1}{2^7 \cdot 3}z^4, \\ S_2(W,z) &= -\frac{1}{2^3 \cdot 3}z^2 p_1 + \frac{5}{2^3 \cdot 3}z^4, \\ S_3(W,z) &= -\frac{1}{2^3}z^2 p_1 + \frac{13}{2^3}z^4. \end{aligned}$$

Note that it is not possible to construct a bounding manifold W of M with a spin structure since then the spin structure on W would induce a spin structure on its boundary, see see Proposition II. 2.15 in [21]. Recall that the spin structure of the bounding manifold was an essential ingredient in the proof for the spin case. Hence, the same method can not be applied to obtain the diffeomorphism and homeomorphism classification in the nonspin case. However, we can use the fact that $L(u^2, u^2) = z^4 \in \mathbb{Q}/\mathbb{Z}$ and an elementary calculation to show that s_2 and the self-linking number determine s_3. We note that this elementary argument can not be used for the spin case.

Lemma 4.5. *If M is a smooth nonspin manifolds of type r and u is a generator of $H^2(M)$. Then*

$$s_3(M) = 3s_2(M) + L(u^2, u^2).$$

Proof. Let (W, z) be a bounding pair of (M, u). Then

$$\begin{aligned} s_3(M) &= -\frac{1}{2^3}z^2 p_1 + \frac{13}{2^3}z^4 = -\frac{1}{2^3}z^2 p_1 + \frac{5}{2^3}z^4 + z^4 \\ &= 3(-\frac{1}{2^3 \cdot 3}z^2 p_1 + \frac{5}{2^3 \cdot 3}z^4) + z^4 = 3s_2(M) + z^4 \ \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

By Corollary 3.3, we know that $L(u^2, u^2)$ and z^4 are the same in $\mathbb{Q}/\mathbb{Z}$. Hence,

$$s_3(M) = 3s_2(M) + L(u^2, u^2).$$

□

Applying Lemma 4.5 to the homeomorphism and diffeomorphism classification, Classification Theorem I, we can replace s_3 by the self-linking number which completes the proof of Theorem B.

5. A Complete Picture of Eschenburg Classification

The Eschenburg space $E_{k,l}$, where $k := (k_1, k_2, k_3)$, $l := (l_1, l_2, l_3) \in \mathbb{Z}^3$ satisfying $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$ and the gcd condition:

$$\begin{aligned} &\gcd(k_1 - l_1, k_2 - l_2) = 1, \ \gcd(k_1 - l_1, k_3 - l_2) = 1,\\ &\gcd(k_2 - l_1, k_1 - l_2) = 1, \ \gcd(k_2 - l_1, k_3 - l_2) = 1,\\ &\gcd(k_3 - l_1, k_1 - l_2) = 1, \ \gcd(k_1 - l_1, k_2 - l_2) = 1, \end{aligned}$$

is defined to be the quotient of a two-sided action of S^1 on $SU(3)$, $E_{k,l} := SU(3)/S^1$, where the S^1-action can be described as follows: $S^1 \times SU(3) \longrightarrow SU(3)$,

$$(z, A) \longmapsto \rho_k(z) A \rho_l(z^{-1}).$$

where $\rho_k(z) = \mathrm{diag}(z^{k_1}, z^{k_2}, z^{k_3})$, $\rho_l(z^{-1}) = \mathrm{diag}(z^{-l_1}, z^{-l_2}, z^{-l_3})$ and $\mathrm{diag} : S^1 \times S^1 \times S^1 \longrightarrow U(3)$ is the embedding defined by

$$(z, v, w) \mapsto \begin{pmatrix} z & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix}.$$

Since the Eschenburg spaces are spin manifolds of odd type with trivial fourth homotopy group, they are classified by Classification Theorem A. Hence, the following classification theorem claimed by B. Kruggel in [20] is now proven.

Theorem 5.1. *Let $E_{k,l}$ and $E_{k',l'}$ be two Eschenburg spaces with the same order of the fourth cohomology group. Let $u \in H^2(E_{k,l})$ and $u' \in H^2(E_{k',l'})$ be both generators. Then*

- *$E_{k,l}$ is (orientation preserving) diffeomorphic to $E_{k',l'}$ if and only if*
$$L(u^2, u^2) = L(u'^2, u'^2), \ s_1(E_{k,l}) = s_1(E_{k',l'}), \ s_2(E_{k,l}) = s_2(E_{k',l'}).$$
- *$E_{k,l}$ is (orientation preserving) homeomorphic to $E_{k',l'}$ if and only if*
$$L(u^2, u^2) = L(u'^2, u'^2), \ p_1(E_{k,l}) = p_1(E_{k',l'}), \ s_2(E_{k,l}) = s_2(E_{k',l'}).$$

In [20], the above invariants of those Eschenburg spaces satisfying condition (C) are computed. A pair (k, l) satisfies condition (C) if the matrix

$$(k_i - l_j) = \begin{pmatrix} k_1 - l_1 & k_1 - l_2 & k_1 - l_3 \\ k_2 - l_1 & k_2 - l_2 & k_2 - l_3 \\ k_3 - l_1 & k_3 - l_2 & k_3 - l_3 \end{pmatrix}$$

contains a row or a column whose entries are pairwise relatively prime. However, B. Kruggel did not know whether or not this condition holds for all Eschenburg spaces.

Unfortunately, from [7], we know that condition (C) is not always satisfied. The homeomorphism and diffeomorphism classification of the Eschenburg spaces not satisfying condition (C) is still an open problem. The following is a complete picture of the classification of the Eschenburg spaces satisfying condition (C).

Theorem 5.2. *For the Eschenburg spaces $E_{k,l}$ satisfying condition (C), the following is a complete set of invariants:*

- *For (orientation preserving) diffeomorphism type,*
  - $|r(k,l)| \in \mathbb{Z}$,
  - $s(k,l)/r(k,l) \in \mathbb{Q}/\mathbb{Z}$,
  - $s_1(k,l) \in \mathbb{Q}/\mathbb{Z}$,
  - $s_2(k,l) \in \mathbb{Q}/\mathbb{Z}$.
- *For (orientation preserving) homeomorphism type,*
  - $|r(k,l)| \in \mathbb{Z}$,
  - $s(k,l)/r(k,l) \in \mathbb{Q}/\mathbb{Z}$,
  - $p_1(k,l)/r(k,l) \in \mathbb{Q}/\mathbb{Z}$,
  - $s_2(k,l) \in \mathbb{Q}/\mathbb{Z}$.

Here

$$
\begin{aligned}
r(k,l) =& \sigma_2(k_1,k_2,k_3) - \sigma_2(l_1,l_2,l_3),\\
s(k,l) =& \sigma_3(k_1,k_2,k_3) - \sigma_3(l_1,l_2,l_3),\\
p_1(k,l) =& 2\sigma_1(k)^2 - 6\sigma_2(k),\\
s_1(k,l) =& \frac{4\cdot|r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)| - (q(k,l))^2}{2^7\cdot 7\cdot r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)}\\
& - s_1(L(k_1-l_1;k_2-l_1,k_3-l_1,k_2-l_2,k_3-l_2))\\
& - s_1(L(k_2-l_1;k_1-l_1,k_3-l_1,k_1-l_2,k_3-l_2))\\
& - s_1(L(k_3-l_1;k_1-l_1,k_2-l_1,k_1-l_2,k_2-l_2)),
\end{aligned}
$$

$$
\begin{aligned}
s_2(k,l) =& \frac{q(k,l)-2}{2^4\cdot 3\cdot r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)}\\
& - s_2(L(k_1-l_1;k_2-l_1,k_3-l_1,k_2-l_2,k_3-l_2))\\
& - s_2(L(k_2-l_1;k_1-l_1,k_3-l_1,k_1-l_2,k_3-l_2))\\
& - s_2(L(k_3-l_1;k_1-l_1,k_2-l_1,k_1-l_2,k_2-l_2)),
\end{aligned}
$$

where σ_i is the i-th elementary symmetric function and

$$\begin{aligned} q(k,l) =&(k_1-l_1)^2+(k_2-l_1)^2+(k_3-l_1)^2+(k_1-l_2)^2 \\ &+(k_2-l_2)^2+(k_3-l_2)^2-(l_1-l_2)^2, \\ s_1(L(p;p_1,p_2,p_3,p_4)) =&\frac{1}{2^5\cdot 7\cdot p}\sum_{k=1}^{|p|-1}\prod_{j=i}^{4} cot(k\pi p_j/p) \\ &+\frac{1}{2^4\cdot p}\sum_{k=1}^{|p|-1}\prod_{j=i}^{4} csc(k\pi p_j/p), \\ s_2(L(p;p_1,p_2,p_3,p_4)) =&\frac{1}{2^4\cdot p}\sum_{k=1}^{|p|-1}(e^{\frac{2\pi ik}{|p|}}-1)\prod_{j=i}^{4} csc(k\pi p_j/p). \end{aligned}$$

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