Topological structure of candidates for positive curvature

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Abstract

In light of recent advances in the study of manifolds admitting Riemannian metrics of positive sectional curvature, the study of certain infinite families of seven dimensional manifolds has become a matter of interest. We determine the cohomology ring structures of manifolds belonging to these families. This particular ring structure indicates the existence of topological invariants distinguishing the corresponding homeomorphism and diffeomorphism type. We show that all families contain representatives of infinitely many homotopy types.

Key words: cohomology rings, cohomogeneity-one actions on manifolds, non-negative sectional curvature, positive sectional curvature

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1. Introduction and background.

Even though Riemannian manifolds admitting positive sectional curvature have been studied for well over 40 years, there are very few known examples of such manifolds. In the recent past, after more than a decade of no discoveries, a new example was found ([8],[12]). This example belongs to one of two infinite families of seven dimensional manifolds which are candidates for positive sectional curvature: it follows from the work of Grove, Verdiani, Wilking and Ziller that any new examples of simply connected, compact cohomogeneity

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one manifolds admitting positive sectional curvature must belong to one of the families $\mathcal{M}$ or $\mathcal{N}$ ([11]).

We now give an overview of the current state of affairs regarding compact, simply connected manifolds admitting metrics of positive sectional curvature. The first major event was the classification of positively curved homogeneous spaces. Carried out between 1961 and 1976, this effort identified the following as being the only compact, simply connected, positively curved homogenous spaces besides the spheres and simply connected projective spaces: one infinite family of seven dimensional manifolds called the Aloff-Wallach spaces and five isolated examples, one each in dimensions seven, six, twelve, thirteen and twenty-four (see [4],[19],[1],[3]).

An additional source of examples are the so-called biquotients. A biquotient is the orbit space of a compact Lie group G under the free action of a closed subgroup $H \subseteq G \times G$. The Eschenburg spaces, an infinite family of simply connected, seven dimensional biquotients of $SU(3)$ under a two-sided circle action, have been extensively studied since their introduction in 1982. All biquotients admit metrics of non-negative sectional curvature, while an infinite subfamily of Eschenburg spaces also admits strictly positive sectional curvature. Two years after the discovery of the Eschenburg spaces, a single compact, simply connected biquotient in dimension six was found to admit positive sectional curvature ([10]). After this, more than a decade passed without any new discoveries. Then, in 1996, another infinite family of compact, simply connected positively curved biquotients was produced, this time in dimension thirteen ([2]).

This comprised all known examples of compact, simply connected, positively curved manifolds prior to 2007, when a member of an infinite family of seven dimensional cohomogeneity one manifolds was discovered to admit positive sectional curvature ([8],[12]). A smooth manifold M is of cohomogeneity one if it supports a smooth action by a Lie group G such that the orbit space M/G is one dimensional. When M is compact and has a finite fundamental group, M/G is diffeomorphic to a closed interval, and M can be described in terms of the acting group G and the isotropy groups of the G-action (see Section 2). The discovered manifold is truly a new example as it is of different homotopy type as all known examples.

This article is concerned with four infinite families of compact, simply connected, seven dimensional cohomogeneity one manifolds, arising from the classification of cohomogeneity one actions on compact, simply connected manifolds in dimensions five through seven ([15]). For convenience, we recall their isotropy group descriptions in Table 1 (cf. [11, Table A], [15, Table I]).

Several considerations motivate our interest in these families. First, the positively curved example of [8] and [12] is a member of the family $\mathcal{M}$. In addition, it follows from the work of Grove, Wilking and Ziller ([11]) that any new examples of simply connected, compact, cohomogeneity one manifolds admitting positive sectional curvature must belong to one of the families $\mathcal{M}$ or $\mathcal{N}$, namely $P_k = M_{(1,1,1+2k,1-2k)}$, $Q_k = N_{(1,1,k,k+1)}$, $R = N_{(3,1,1,2)}$ are their candidates for positive sectional curvature. Moreover, the positively curved Eschenburg spaces correspond to the subfamily $\{O_{(p,p+1:2)}\} \subseteq \mathcal{O}$ ([7],[10],[11]). It is not known in general, however, whether members of the family $\mathcal{O}$ admit metrics of non-negative sectional curvature. In contrast, all members of the families $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ are known to admit non-negative sectional curvature ([13, Theorem E]). Furthermore, these families contain several subfamilies that carry 3-Sasakian structures, namely the P_k , Q_k and the Eschenburg spaces $E_{(a,b,c,a+b+c,0,0)}$. Finally, these four families comprise the full class of seven dimensional primitive cohomogeneity one manifolds ([15]). A cohomogeneity one manifold is non-primitive if there is a G-equivariant map $M \longrightarrow G/L$ for some proper subgroup $L \subset G$ (see [11]). Otherwise the G-action is called primitive. Any non-primitive cohomogeneity one manifold is diffeomorphic to the total space of a bundle over a homogenous space whose fiber is a primitive manifold ([15]). Note that the classification given in [15] is purely algebraic; the underlying topological structure of these manifolds is largely unknown. In the following we give a complete description of the cohomology rings of all four families. Note that manifolds are assumed to be without boundary unless explicitly stated otherwise. Also unless otherwise stated, cohomology groups take their coefficients in the integers.

Finally, we denote by $\mathbb{Z}_r$ the cyclic group of order r, and understand $\mathbb{Z}_0$ to be infinite cyclic and $\mathbb{Z}_1$ to be trivial.

Theorem 1.1. *A compact, simply connected, seven dimensional, primitive cohomogeneity one manifold M is a member of:*

a) the subfamily of $\mathcal{L}$ with the parameter p_+ odd and $p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0$, or:

b) the family $\mathcal{N}$, or:

c) the subfamily of $\mathcal{O}$ with $|p|$ and $|q|$ not both equal to one

if and only if the cohomology groups of M are given by:

$$H^k(M) \cong \begin{cases} \mathbb{Z} & k = 0, 2, 5, 7 \\ \mathbb{Z}_r, \ r \neq 0, 1 & k = 4 \\ 0 & \text{otherwise.} \end{cases}$$

where $r = \frac{1}{4}|p_+^2 q_-^2 - p_-^2 q_+^2|$ for $M \in \mathcal{L}$, $r = |p_+^2 q_-^2 - p_-^2 q_+^2|$ for $M \in \mathcal{N}$ and $r = |p^2 - q^2|$ for $M \in \mathcal{O}$. Furthermore, the cohomology ring of any of these manifolds is completely generated (under the cup product) by cohomology group generators $x \in H^2(M)$ and $y \in H^5(M)$.

Theorem 1.2. *A compact, simply connected, seven dimensional, primitive cohomogeneity one manifold M is a member of the subfamily of $\mathcal{L}$ with the parameter p_+ even, if and only if the cohomology groups of M are given by:*

$$H^k(M) \cong \begin{cases} \mathbb{Z} & k = 0, 2, 7 \\ \mathbb{Z}_2 & k = 3 \\ \mathbb{Z}_r, \ r \neq 0, 1 & k = 4 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & k = 5 \\ 0 & \text{otherwise.} \end{cases}$$

where $r = |p_+^2 q_-^2 - p_-^2 q_+^2|$. Furthermore, if the class x generates $H^2(M)$ and y generates the free part of $H^5(M)$, then x^2 generates $H^4(M)$ and xy generates $H^7(M)$.

Several corollaries follow from Theorem 1.1. The first determines the subfamilies for which the Kreck-Stolz invariants exist:

Corollary 1.3. *A compact, simply connected, seven dimensional, primitive cohomogeneity one manifold admits a Kreck-Stolz invariant if and only if:*

a) it is a member of the subfamily of $\mathcal{L}$ *with the parameter* p_+ *odd and* $p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0$*, or:*

b) it is a member of the family $\mathcal{N}$*, or:*

c) it is a member of the subfamily $\mathcal{O}$ *with* $|p|$ *and* $|q|$ *not both equal to one.*

A comparison with the cohomology ring of the Eschenburg spaces yields:

Corollary 1.4. *A compact, simply connected, seven dimensional, primitive cohomogeneity one manifold has the cohomology ring of an Eschenburg space if and only if:*

a) it is any member of the family $\mathcal{N}$ *or:*

b) it is a member of the family $\mathcal{O}$ *and one of the parameters* p *or* q *is even.*

By examination of the order of the fourth cohomology group, it is apparent that:

Corollary 1.5. *Every family of compact, simply connected, seven dimensional, primitive cohomogeneity one manifolds contains representatives of infinitely many distinct homotopy types.*

To outline of the remainder of this article: Section 2 opens with a brief discussion of the relationship between the isotropy groups of a cohomogeneity one action and the topology of a manifold, motivating the introduction of double disk bundles. Table 1 appears, summarizing the isotropy group description of the four families of manifolds under consideration. Next, two long exact cohomology sequences are described. Finally, Lemma 2.2 gives a way of identifying certain cohomology ring generators for those double disk bundles which fulfill the stated cohomological conditions.

The proofs of Theorems 1.1 and 1.2 then proceed in tandem. Cohomology groups are computed in Section 3. The order of the fourth cohomology groups are expressed in terms of the parameters of the principal isotropy groups of the cohomogeneity one actions. Restrictions on the parameters under which the fourth cohomology groups are guaranteed to be both non-trivial and finite cyclic are found. In Section 4, cohomology ring generators (under the cup product) are determined, completing the proofs.

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2. Computing cohomology for double disk bundles.

When a compact, connected, smooth manifold M has a finite fundamental group, a cohomogeneity one action by a compact Lie group G induces additional topological structure which allows for a particularly nice description of the manifold in terms of the isotropy groups of the G-action. In this case, M/G is diffeomorphic to a closed interval. The G-orbits over the interval's interior points are called principal orbits, while those over the endpoints are called non-principal. Up to conjugation in G, there are three isotropy groups of the G-action: a principal isotropy group H and two non-principal isotropy groups K_- and K_+. The non-principal orbits are diffeomorphic to $G/K_\pm$, while a principal orbit is diffeomorphic to G/H.

In fact, these groups are sufficient to describe such a manifold. A compact, connected manifold M with finite fundamental group supports a cohomogeneity one action by a compact Lie group G if and only if there are closed subgroups and inclusions $H \subseteq K_-, K_+ \subseteq G$ such that $K_\pm/H$ are diffeomorphic to spheres $S^{t_\pm - 1}$. Then M is diffeomorphic to the union of the total spaces $D(G/K_\pm) := G \times_{K_\pm} D^{t_\pm}$ of disk bundles equivariantly identified along their common boundary $G \times_{K_\pm} S^{t_\pm - 1}$. A principal orbit G/H of the G-action is diffeomorphic to the total space $G \times_{K_\pm} (K_\pm/H) \approx G \times_{K_\pm} S^{t_\pm - 1}$ of the boundary sphere bundles of the disk bundles. The non-principal orbits $G/K_\pm$ are diffeomorphic to the base spaces of the disk bundles, and are identified with the zero sections in their respective total spaces $D(G/K_\pm)$. For further details, refer to [6], [13] and [17].

All members of the families $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{O}$ are compact and simply connected, and all admit a cohomogeneity one action by $G = S^3 \times S^3$. We regard S^3 as the group of unit quaternions. The parameters p and q of a circle group $\{(e^{ip\theta}, e^{iq\theta})\} \subset S^3 \times S^3$ are necessarily relatively prime integers. Table 1 lists the remaining data needed to describe the manifolds; namely, the isotropy groups together with restrictions on the parameters that ensure an embedding of the principal isotropy group H in the non-principal isotropy groups $K_\pm$. For example, in the description of the family $\mathcal{L}$ the principal isotropy group $H = \langle (i,i) \rangle$ is the cyclic group of order four generated by the diagonal embedding of the unit quaternion i (the notation $\langle q_1, \ldots, q_n \rangle$ denoting the subgroup generated by the elements $q_1, \ldots, q_n$), the non-principal isotropy group $K_+ = \{(e^{jp_+\theta}, e^{jq_+\theta})\} \cdot H$ is the group whose elements are products of an element of the circle group with an element of H, and the congruence of the parameters $p_-, q_- \equiv 1 \bmod 4$ of the non-principal isotropy group $K_- = \{(e^{ip_-\theta}, e^{iq_-\theta})\}$ ensures that H is a subgroup. Note that our notation is consistent with [11, Chapter 13].

The description of a compact, simply connected cohomogeneity one manifold as the union of two disk bundles is an example of a more general topological construction, which we call a double disk bundle. This is a quotient space $X = D(B_-) \cup_\varphi D(B_+)$ where the two spaces $D(B_\pm)$ are the total spaces of disk bundles over paracompact bases $B_\pm$ and the attaching map φ is a homeomorphism of the boundaries $\partial D(B_\pm)$. In the case of a cohomogeneity one manifold, the spaces $B_\pm$ are the non-principal orbits $G/K_\pm$, the boundaries $\partial D(G/K_-) = \partial D(G/K_+) = G/H$, and the attaching map φ is the identity map. So for M cohomogeneity one, $M = D(G/K_-) \cup_{id} D(G/K_+)$. Note that one can also glue with any G-equivariant map instead of the identity. The number of manifolds (up to equivariant diffeomorphism) that one obtains in this fashion is controlled by the Neumann groups, as defined in [18].

2.1. Two long exact cohomology sequences.

The cohomology groups of a double disk bundle $X = D(B_-) \cup_\varphi D(B_+)$ can be computed in terms of the disk bundles $D(B_\pm) \to B_\pm$ using the Mayer-

Family	Isotropy groups $H \subseteq K_-, K_+$ / restrictions on parameters
$\mathcal{L} :=$ $\{L_{(p_-,q_-),(p_+,q_+)}\}$	$\langle(i,i)\rangle \subseteq \{(e^{ip_-\theta}, e^{iq_-\theta})\}$, $\{(e^{jp_+\theta}, e^{jq_+\theta})\} \cdot H$
	$p_-, q_- \equiv 1 \bmod 4$
$\mathcal{M} :=$ $\{M_{(p_-,q_-),(p_+,q_+)}\}$	$\Delta Q \subseteq \{(e^{ip_-\theta}, e^{iq_-\theta})\} \cdot H$, $\{(e^{jp_+\theta}, e^{jq_+\theta})\} \cdot H$
	ΔQ the diagonal embedding of $\langle 1, i, j, k\rangle$; $p_\pm, q_\pm \equiv 1 \bmod 4$
$\mathcal{N} :=$ $\{N_{(p_-,q_-),(p_+,q_+)}\}$	$\langle(h_1,h_2),(1,-1)\rangle \subseteq \{(e^{ip_-\theta}, e^{iq_-\theta})\} \cdot H$, $\{(e^{jp_+\theta}, e^{jq_+\theta})\} \cdot H$
	$h_1, h_2 \in \{i, -i\}$ with signs chosen so that (h_1,h_2) lies in $\{(e^{ip_-\theta}, e^{iq_-\theta})\}$; p_- and $q_\pm$ odd, p_+ even
$\mathcal{O} :=$ $\{O_{(p,q:m)}\}$	$\mathbb{Z}_m \subseteq \{(e^{ip\theta}, e^{iq\theta})\}$, $\Delta S^3 \cdot H$
	either $m = 1$ (with no restrictions on p or q) or $m = 2$ and p is even

Table 1: Isotropy groups description of compact, simply connected, seven dimensional, primitive cohomogeneity one manifolds.

Vietoris sequence or the long exact sequences of the pairs $(X, B_\pm)$. If the attaching map φ is the identity map, the Mayer-Vietoris sequence can be modified. In the case of a cohomogeneity one manifold, this modified sequence relates the cohomology of the manifold to the cohomology of the principal and non-principal orbits.

Denote by $\partial D(B)$ the common boundary of the total spaces $D(B_\pm)$ of the disk bundles. Restricting the projections of the disk bundles to their boundaries results in sphere bundles $\partial D(B) \to B_\pm$. Let $\pi_\pm$ denote these restricted projections, and define a homomorphism $\pi^* := \pi_-^* - \pi_+^*$. Finally, note that the deformation retractions of $D(B_\pm)$ onto $B_\pm$ induce isomorphisms of the cohomology groups. Making the appropriate substitutions in the Mayer-Vietoris cohomology sequence gives the long exact sequence:

$$\cdots \to H^k(X) \xrightarrow{\psi} H^k(B_-) \oplus H^k(B_+) \xrightarrow{\pi^*} H^k(\partial D(B)) \xrightarrow{\delta} H^{k+1}(X) \to \cdots \quad (1)$$

where ψ is the composition of the homomorphisms induced by the inclusions of $D(B_\pm)$ in X with the deformation retractions of $D(B_\pm)$ onto $B_\pm$. The homomorphism δ is the boundary homomorphism of the Mayer-Vietoris sequence (cf. [13, Sequence 3.4]).

The long exact sequence of the pair (X, B_+) can also be modified, assuming the disk bundle $D^t \hookrightarrow D(B_-) \to B_-$ is orientable; that is, if the structure group $O_t(\mathbb{R})$ can be reduced to $SO_t(\mathbb{R})$. When applied to a cohomogeneity one manifold, the resulting sequence relates the cohomology of the manifold to the cohomology of the principal orbits.

Begin with the long exact cohomology sequence of the pair (X, B_+). Since the inclusion $B_+ \hookrightarrow D(B_+)$ is a homotopy equivalence, the inclusion of pairs $(X, B_+) \hookrightarrow (X, D(B_+))$ induces isomorphisms of the relative cohomology groups $H^k(X, B_+) \cong H^k(X, D(B_+))$ by the five lemma. By excision, there are isomorphisms $H^k(X, D(B_+)) \cong H^k(D(B_-), \partial D(B))$, and orientability of the bundle $D(B_-) \to B_-$ guarantees the existence of a Thom isomorphism from $H^{k-t}(B_-)$ to $H^k(D(B_-), \partial D(B))$. This gives an isomorphism $H^{k-t}(B_-) \cong H^k(X, B_+)$.

Because the bundle projection $D(B_-) \to B_-$ followed by the inclusion $B_- \hookrightarrow X$ is homotopic to the inclusion $D(B_-) \hookrightarrow X$, the composition of the Thom isomorphism with the inverse of the excision isomorphism is an $H^*(X)$-module homomorphism from $H^{k-t}(B_-)$ to $H^k(X, D(B_+))$ ([9, Corollary 11.20]). Since homomorphisms induced on cohomology by topological maps respect cup products, the group isomorphism from $H^k(X, D(B_+))$ to $H^k(X, B_+)$ is also an $H^*(X)$-module isomorphism. Thus, there is an $H^*(X)$-module isomorphism from $H^{k-t}(B_-)$ to $H^k(X, B_+)$. Define J to be the composition of this isomorphism with the homomorphism j_+^* from $H^k(X, B_+)$ to $H^k(X)$ in the long exact sequence of the pair (X, B_+). Checking the definition of j_+^*, one sees that it is a homomorphism of $H^*(X)$-modules. Thus, J is an $H^*(X)$-module homomorphism. This will be key in identifying cohomology ring generators.

Making the appropriate substitutions in the sequence of the pair (X, B_+), we have the long exact sequence:

$$\cdots \to H^{k-t}(B_-) \xrightarrow{J} H^k(X) \xrightarrow{i_+^*} H^k(B_+) \xrightarrow{\delta} H^{k-t+1}(B_-) \to \cdots \tag{2}$$

(cf. [14, Sequences 4.1.a, 4.1.b]). An analogous sequence, with the roles of B_+ and B_- reversed, exists if the bundle $D(B_+) \to B_+$ is orientable. Furthermore, such a sequence exists for any bundle, regardless of orientability, if integral coefficients are replaced by $\mathbb{Z}_2$-coefficients.

If B is a closed orientable submanifold of an orientable manifold M, then it is known that the normal disk sub-bundle over B in the tangent bundle of M is an orientable bundle ([5, p.66]). As we shall see, both non-principal orbits for manifolds in the family $\mathcal{O}$ are orientable, while only the orbit G/K_- is orientable for members of $\mathcal{L}$ and $\mathcal{N}$. Since the orbits are closed submanifolds of a simply connected manifold, at least one long exact sequence of this type exists for each of these manifolds. Note that both non-principal orbits of members of $\mathcal{M}$ are non-orientable (see [11]).

2.2. Two lemmas and a commutative diagram.

We now derive two lemmas from Sequences 1 and 2, and introduce a commutative ladder of long exact sequences. These, together with the sequences, are the main tools on which we rely in the proofs of Theorems 1.1 and 1.2. Both lemmas apply to double disk bundles, and require orientability of at least one of the bundles.

Lemma 2.1 is used in the proofs of Theorems 1.1 and 1.2 to conclude that the fourth cohomology groups of manifolds in the families $\mathcal{L}$ and $\mathcal{O}$ are cyclic:

Lemma 2.1. *Let $X = D(B_-) \cup_{id} D(B_+)$ be a double disk bundle where the bundle $D^t \hookrightarrow D(B_-) \to B_-$ is orientable. For a fixed integer κ, suppose $H^{\kappa-t}(B_-)$ is cyclic and both groups $H^{\kappa}(B_\pm)$ are trivial. Furthermore, suppose $H^{\kappa-1}(\partial D(B))$ is finitely generated and free, and has the same rank as the free part of $H^{\kappa-1}(B_-) \oplus H^{\kappa-1}(B_+)$. Let $r \geq 0$ be the absolute value of the determinant of the restriction of the homomorphism π^* to the free part of $H^{\kappa-1}(B_-) \oplus H^{\kappa-1}(B_+)$. Then $H^{\kappa}(X)$ is the cyclic group of order r.*

Proof. Setting $k = \kappa$ in Sequence 2, one sees that $H^\kappa(X)$ must be a cyclic group. Examination of the section of Sequence 1 for which $k = \kappa - 1$ and $k = \kappa$ discloses that $H^\kappa(X)$ is equal to the cokernel of the restriction of π^* to the free part of $H^{\kappa-1}(B_-) \oplus H^{\kappa-1}(B_+)$. Then the Smith normal form can be used to relate the determinant of the restriction of π^* to the order of $H^\kappa(X)$. $\square$

Lemma 2.2 is used in the proofs of Theorems 1.1 and 1.2 to identify generators of the cohomology rings:

Lemma 2.2. *Let $X = D(B_-) \cup_\varphi D(B_+)$ be a double disk bundle over a connected base, where the disk bundle $D^t \hookrightarrow D(B_-) \to B_-$ is orientable. Suppose $H^t(X)$ is infinite cyclic, and $H^t(B_+)$ is finite cyclic of order $n \geq 1$. Let $i^*_\pm$ be the homomorphisms induced on cohomology by the inclusions of $B_\pm$ in X, and suppose $i^*_+ : H^t(X) \to H^t(B_+)$ is a surjection. Finally, suppose κ is a fixed integer, $\kappa > t$, such that the following hold:*

1. *$H^\kappa(X)$ is a non-trivial cyclic group and $H^\kappa(X) \xrightarrow{i^*_+} H^\kappa(B_+)$ is the zero homomorphism.*
2. *$H^{\kappa-t}(B_-) \cong \mathbb{Z} \cdot \gamma \oplus T$ where T is torsion and the free part is generated by γ. If $H^\kappa(X)$ is finite, the orders of elements of T are relatively prime to the order of $H^\kappa(X)$.*
3. *There exists a class α in $H^{\kappa-t}(X)$ with image $i^*_-(\alpha) = s\gamma + \beta$ (for $\beta \in T$) such that: if $H^\kappa(X)$ is free, then $|s| = n$; otherwise, s is relatively prime to the order of $H^\kappa(X)$.*

Then the cohomology class $x \smile \alpha$ generates $H^\kappa(X)$, where x is a generator of $H^t(X)$.

Proof. Let $\mathbb{1}_-$ be the unit of the cohomology ring $H^*(B_-)$. Setting $k = t$ in Sequence 2, and assuming the hypotheses regarding $H^t(X)$ and $H^t(B_+)$ hold, one has a short exact sequence:

$$0 \to H^0(B_-) \cong \mathbb{Z} \xrightarrow{J} H^t(X) \cong \mathbb{Z} \xrightarrow{i^*_+} H^t(B_+) \cong \mathbb{Z}_n \to 0.$$

From this we conclude that the homomorphism J from $H^0(B_-)$ to $H^t(X)$ is multiplication by n, and $J(\mathbb{1}_-) = \pm nx$.

Now, let $k = \kappa$ in Sequence 2. By Condition 1, the homomorphism from $H^\kappa(X)$ to $H^\kappa(B_+)$ is the zero homomorphism, hence by exactness the homomorphism J from $H^{\kappa-t}(B_-) \cong \mathbb{Z}\cdot\gamma \oplus T$ to $H^\kappa(X)$ is a surjection. Torsion elements of $H^{\kappa-t}(B_-)$ are in the kernel of J (by Condition 2), so $J(\gamma)$ generates $H^\kappa(X)$.

We now consider separately the case in which $H^\kappa(X)$ is infinite cyclic, and that in which it is finite cyclic. First, suppose $H^\kappa(X)$ is infinite cyclic. Let $i_-^*(\alpha) = \pm n\gamma + \beta \in H^{\kappa-t}(B_-)$ where β is torsion, as required by Condition 3. The homomorphism J maps the torsion element β to zero in $H^\kappa(X)$, hence:

$$nJ(\gamma) = \pm J(\pm n\gamma) \pm J(\beta) = \pm J(\pm n\gamma + \beta) = \pm J(i_-^*(\alpha)) = \pm J(\mathbb{1}_- \smile i_-^*(\alpha))$$

where the negative signs correspond to the case in which $i_-^*(\alpha) = -n\gamma + \beta$. Recall that J is an $H^*(X)$-module homomorphism, so:

$$J(\mathbb{1}_- \smile i_-^*(\alpha)) = J(\mathbb{1}_-) \smile \alpha = \pm n(x \smile \alpha).$$

Since the generator $J(\gamma)$ is non-trivial in $H^\kappa(X) \cong \mathbb{Z}$ and $n \neq 0$, cancellation implies that $x \smile \alpha = \pm J(\gamma)$; so $x \smile \alpha$ generates $H^\kappa(X)$.

On the other hand, suppose $H^\kappa(X)$ is finite cyclic. Let $i_-^*(\alpha) = s\gamma + \beta$ where s is relatively prime to the order of $H^\kappa(X)$, thus satisfying Condition 3. A calculation similar to the one carried out in the previous case shows that $sJ(\gamma) = \pm n(x \smile \alpha)$. The class $J(\gamma)$ generates $H^\kappa(X)$, and the order of $H^\kappa(X)$ is relatively prime to s, therefore $sJ(\gamma) = \pm n(x \smile \alpha)$ also generates $H^\kappa(X)$. But if a multiple of $x \smile \alpha$ generates a finite cyclic group, then $x \smile \alpha$ itself must be a generator. □

A useful tool for determining whether Condition 3 of Lemma 2.2 holds is the commutative ladder of long exact sequences:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{j_-^*} & H^k(X) & \xrightarrow{i_-^*} & H^k(B_-) & \xrightarrow{\delta_-} & H^{k+1}(X, B_-) & \xrightarrow{j_-^*} & \cdots \\
 & & \downarrow{\scriptstyle i_+^*} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\
\cdots & \xrightarrow{j^*} & H^k(B_+) & \xrightarrow{i^*} & H^k(\partial D(B)) & \xrightarrow{\delta} & H^{k+1}(D(B_+), \partial D(B)) & \xrightarrow{j^*} & \cdots .
\end{array} \tag{3}$$

To construct this diagram, begin with the long exact sequence of the pair (X, B_-), which forms the top row. Consider the commutative ladder of the long exact sequences of the pairs (X, B_-) and $(X, D(B_-))$ induced by the inclusion of pairs $(X, B_-) \hookrightarrow (X, D(B_-))$. All vertical homomorphisms in this ladder are isomorphisms; the inclusion of B_- in $D(B_-)$ is a homotopy equivalence, so the five lemma applies to the relative groups. To complete the diagram, use the commutative ladder of long exact sequences induced by the inclusion of pairs $(D(B_+), \partial D(B)) \hookrightarrow (X, D(B_-))$. Note that the homomorphism of the relative cohomology groups induced by this inclusion of pairs is the excision isomorphism. This gives a commutative ladder between the long exact sequences of the pairs (X, B_-) and $(D(B_+), \partial D(B))$ in which the vertical homomorphisms between the relative cohomology groups are isomorphisms. Finally, the inclusion of B_+ in $D(B_+)$ is a homotopy equivalence, so the cohomology groups of $D(B_+)$ may be replaced with those of B_+.

3. Cohomology groups.

In this section, we compute the cohomology groups of members of the families $\mathcal{L}$ and $\mathcal{O}$. We also recount the cohomology groups of members of the families $\mathcal{M}$ and $\mathcal{N}$, originally found in [11]. We express the order of the fourth cohomology groups in terms of the parameters of the principal isotropy groups of the cohomogeneity one actions, identify restrictions on the parameters guaranteeing that these groups are both non-trivial and finite, and indicate when they have odd order.

Observe that all manifolds in question are compact, simply connected and seven dimensional ([15]). Because they are simply connected, they are orientable. Easy calculations using Poincaré duality together with the universal coefficient theorem show that they have infinite cyclic cohomology in dimensions zero and seven and trivial cohomology in dimensions one and six. It remains to find the second through fifth cohomology groups.

3.1. Cohomology groups of the family $\mathfrak{L}$.

Recall that this family is described by the groups:

$$H = \langle (i,i) \rangle \subseteq K_- = \{(e^{ip_-\theta}, e^{iq_-\theta})\}, K_+ = \{(e^{jp_+\theta}, e^{jq_+\theta})\} \cdot H \subseteq G = S^3 \times S^3$$

where p_-, q_- and p_+, q_+ are pairs of relatively prime integers, and p_- and q_- are both congruent to 1 modulo 4. This family naturally splits into two subfamilies, depending on whether p_+ is even or odd. The cohomology of the principal orbit G/H and the non-principal orbit G/K_- is the same in both cases. The principal orbit $G/H = S^3 \times S^3/\langle(i,i)\rangle$ is homeomorphic to the product $S^3 \times (S^3/\langle i \rangle)$ of the 3-sphere with the lens space $S^3/\langle i \rangle \approx L_4(1,1)$. An explicit homeomorphism is given by $[q_1, q_2] \mapsto (q_1 {q_2}^{-1}, [q_2])$. The non-principal orbit $G/K_- = S^3 \times S^3/\{(e^{ip_-\theta}, e^{iq_-\theta})\}$ is always homeomorphic to $S^3 \times S^2$ by [20, Proposition 2.3]. The orbit G/K_+, however, varies depending on the parity of p_+.

Case 1. Suppose p_+ is odd. Then the cohomology groups of the non-principal orbits G/K_+ were calculated in [11, Lemma 13.3a], where they were shown to be:

$$H^k(G/K_+) \cong \begin{cases} \mathbb{Z} & k = 0,3 \\ \mathbb{Z}_2 & k = 2,5 \\ 0 & \text{otherwise.} \end{cases}$$

Let $L := L_{(p_-,q_-),(p_+,q_+)}$ be a member of the subfamily of $\mathfrak{L}$ with p_+ odd. We know that $H^0(L) \cong H^7(L) \cong \mathbb{Z}$. The orbit $G/K_- \approx S^3 \times S^2$ is a closed orientable submanifold of codimension 2, so the normal disk bundle over G/K_- is an orientable bundle with fiber D^2. Setting $t = 2$, $\kappa = 4$, $B_\pm = G/K_\pm$ and $\partial D(B) = G/H$, it follows from Lemma 2.1 that $H^4(L) \cong \text{coker}\pi^* \cong \mathbb{Z}_r$. Recall that r is (up to sign) the determinant of the homomorphism π^* from the rank two free abelian group $H^3(G/K_-) \oplus H^3(G/K_+) \cong \mathbb{Z} \oplus \mathbb{Z}$ to the rank two free abelian group $H^3(G/H) \cong \mathbb{Z} \oplus \mathbb{Z}$. Apply Sequences 1 and 2 (taking $t = 2$) to find the remaining cohomology groups:

$$H^k(L) \cong \begin{cases} \mathbb{Z} & k = 0, 2, 5, 7 \\ \ker\pi^* & k = 3 \\ \mathbb{Z}_r & k = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $H^3(L) \cong \ker\pi^*$ will be trivial if and only if $\det(\pi^*) \neq 0$.

To find $r = |\det(\pi^*)|$, we follow the example of [13, Proposition 3.3]. Consider the diagram:

$$\begin{array}{ccc} H^3(G) \cong \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{\tau^* = \tau_-^* - \tau_+^*} & H^3(G/K_-^\circ) \oplus H^3(G/K_+^\circ) \cong \mathbb{Z} \oplus \mathbb{Z} \\ \uparrow \eta^* & & \uparrow \mu^* = \mu_-^* \times \mu_+^* \\ H^3(G/H) \cong \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{\pi^* = \pi_-^* - \pi_+^*} & H^3(G/K_-) \oplus H^3(G/K_+) \cong \mathbb{Z} \oplus \mathbb{Z} \end{array} \tag{4}$$

where the homomorphisms $\tau_\pm^*$ and η^* are induced by orbit maps, and $\mu_\pm^*$ are the homomorphisms induced by the maps $gK_\pm^\circ \mapsto gK_\pm$ (which are themselves induced by the inclusions of the identity components $K_\pm^\circ$ in $K_\pm$). In the present case, K_- is connected; hence, $K_- = K_-^\circ$ and μ_-^* is the identity. And since μ_+^* is an isomorphism by [11, Lemma 13.3a], we have $|\det(\mu^*)| = 1$.

We next wish to find $|\det(\eta^*)|$. Recall that G/H is homeomorphic to $S^3 \times (S^3/\langle i \rangle)$. Uniqueness of the universal cover implies that the composition $S^3 \times S^3 \xrightarrow{\eta} G/H \xrightarrow{\cong} S^3 \times (S^3/\langle i \rangle)$ induces a commutative square on cohomology:

$$\begin{array}{ccc} H^3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{\cong} & H^3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z} \\ \uparrow \eta^* & & \uparrow (id_{S^3} \times f)^* \\ H^3(G/H) \cong \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{\cong} & H^3(S^3 \times S^3/\langle i \rangle) \cong \mathbb{Z} \oplus \mathbb{Z} \end{array} \tag{5}$$

where f is the projection of universal cover of $S^3/\langle i \rangle$ by S^3. An argument involving the Künneth isomorphism shows that there are bases for $H^3(S^3 \times S^3/\langle i \rangle)$ and $H^3(S^3 \times S^3)$ such that $(id_{S^3} \times f)^* = {id_{S^3}}^* \times f^*$. The covering degree $deg(f) = \pm 4$ implies that $|\det(\eta^*)| = |\det({id_{S^3}}^* \times f^*)| = 4$.

The determinant of τ^* follows from [13, Proposition 3.3]. They find a basis of $H^3(S^3 \times S^3)$ with respect to which $\operatorname{im} \tau^*_\pm = \langle(-q^2_\pm, p^2_\pm)\rangle$. Hence, $|\det(\tau^*)| = |\det(\tau^*_- - \tau^*_+)| = |p^2_+ q^2_- - p^2_- q^2_+|$. We conclude that the absolute value of the determinant of π^* is $|\det(\pi^*)| = |\det({\eta^*}^{-1})||\det(\tau^*)||\det(\mu^*)| = \frac{1}{4}|p^2_+ q^2_- - p^2_- q^2_+|$.

In this family, the parameters p_+ and q_+ are odd and $p_-, q_- \equiv 1 \bmod 4$. If we set $p_+ = 2k+1$, $q_+ = 2l+1$, $p_- = 4m+1$ and $q_- = 4n+1$ for some integers k, l, m and n, we see that the parity of $r = \frac{1}{4}|p^2_+ q^2_- - p^2_- q^2_+|$ agrees with the parity of $k(k+1) - l(l+1)$, which must be even. It follows that $H^4(L) \cong \mathbb{Z}_r$ is a non-trivial cyclic group of even order. Thus, $H^3(L)$ is the trivial group and $H^4(L)$ is a non-trivial finite cyclic group of even order if and only if $|p^2_+ q^2_- - p^2_- q^2_+| \neq 0$.

Case 2. On the other hand, suppose p_+ is even. Let $K' = \{(e^{jp_+\theta}, e^{jq_+\theta})\} \cdot \langle(1, -1), (i, i)\rangle$ be a subgroup of $S^3 \times S^3$. Because p_+ is even, the inclusion of $K_+ = \{(e^{jp_+\theta}, e^{jq_+\theta})\} \cdot \langle(i, i)\rangle$ in K' as a subgroup induces a continuous bijection from the compact space G/K_+ to the Hausdorff space G/K'. It follows that G/K_+ is homeomorphic to G/K'. Thus, the cohomology of G/K_+ is the same as that of G/K', which was shown in [11, Lemma 13.6b] to be:

$$H^k(G/K_+) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}_4 & k = 2 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & k = 3 \\ \mathbb{Z}_2 & k = 5 \\ 0 & \text{otherwise.} \end{cases}$$

Let $L := L_{(p_-,q_-),(p_+,q_+)}$ be a member of the subfamily of $\mathcal{L}$ with p_+ even. Using Sequence 1, Poincaré duality and the universal coefficient theorem, we find that $H^0(L) = H^2(L) = H^7(L) \cong \mathbb{Z}$ and $H^5(L) = \mathbb{Z} \oplus \mathbb{Z}_2$. Similarly, $H^3(L)$ and $H^4(L)$ are, respectively, the kernel and cokernel of the homomorphism $\pi^* = \pi^*_- - \pi^*_+$ from $H^3(G/K_-) \oplus H^3(G/K_+)$ to $H^3(G/H)$ in Sequence 1. Applying Lemma 2.1 with $t = 2$ and $\kappa = 4$, $H^4(L)$ is cyclic of order $r = |\det(\pi^*|_{\mathbb{Z}\oplus\mathbb{Z}})|$. In this case, there is a diagram:

$$\begin{array}{ccc} H^3(G) \cong \mathbb{Z}\oplus\mathbb{Z} & \xleftarrow{\tau^*=\tau^*_- - \tau^*_+} & H^3(G/K_-^\circ)\oplus H^3(G/K_+^\circ)\cong \mathbb{Z}\oplus\mathbb{Z} \\ \uparrow \eta^* & & \uparrow \mu^*=\mu^*_-\times\mu^*_+ \\ H^3(G/H)\cong\mathbb{Z}\oplus\mathbb{Z} & \xleftarrow{\pi^*=\pi^*_- - \pi^*_+} & H^3(G/K_-)\oplus H^3(G/K_+)\cong \mathbb{Z}\oplus(\mathbb{Z}\oplus\mathbb{Z}_2). \end{array}$$

Comparing this to Diagram 4, we see that the homomorphisms η^*, τ^* and μ^*_- are the same. By [11, Lemma 13.6], the homomorphism μ^*_+ is multiplication by ± 4 on the free part of $H^3(G/K_+)$, while the $\mathbb{Z}_2$ summand is clearly in the kernel. We conclude that $r = |p_+^2 q_-^2 - p_-^2 q_+^2|$. Since p_+ is even while $q_\pm$ and p_- are odd, r is always odd, so $H^4(L)$ is finite. Also, $r = |(p_+q_- + p_-q_+)(p_+q_- - p_-q_+)| \neq 1$ since the parameters $p_\pm$ and $q_\pm$ are non-zero. This can be confirmed by checking the four possible cases for the equation $r = |(a+b)(a-b)| = 1$ where a and b are integers. Hence, $H^4(L)$ is a non-trivial finite cyclic group of odd order; and by Poincaré duality and the universal coefficient theorem, $H^3(L) \cong \mathbb{Z}_2$. Thus, the cohomology groups of L are:

$$H^k(L) \cong \begin{cases} \mathbb{Z} & k = 0,2,7 \\ \mathbb{Z}_2 & k = 3 \\ \mathbb{Z}_r & k = 4 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & k = 5 \\ 0 & \text{otherwise.} \end{cases}$$

3.2. Cohomology groups of the family $\mathcal{M}$.

The topology of the family $\mathcal{M}$ is described in [11, Theorem 13.1]. A member M of this family has non-trivial cohomology groups $H^0(M) = H^7(M) \cong \mathbb{Z}$ and $H^4(M) \cong \mathbb{Z}_r$ a finite group of order $r = \frac{1}{8}|p_-^2 q_+^2 - p_+^2 q_-^2|$ whenever $p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0$. Otherwise, the non-trivial cohomology groups are $H^0(M) = H^3(M) = H^4(M) = H^7(M) \cong \mathbb{Z}$. Note these manifolds do not admit Kreck-Stolz invariants.

3.3. Cohomology groups of the family $\mathcal{N}$.

The cohomology groups of a member $N := N_{(p_-,q_-),(p_+,q_+)}$ of this family, as computed in [11, Theorem 13.5], are:

$$H^k(N) \cong \begin{cases} \mathbb{Z} & k = 0,2,5,7 \\ \mathbb{Z}_r & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

where the order of the cyclic group $H^4(N)$ is $r = |p_-^2 q_+^2 - p_+^2 q_-^2|$. Since p_+ is required to be even while p_-, q_- and q_+ are odd, r must be odd and (as was the case of the family $\mathcal{L}$ for p_+ even) cannot equal one. Thus, $H^4(N)$ is a non-trivial finite cyclic group of odd order.

3.4. Cohomology groups of the family $\mathcal{O}$.

Recall that this family is described by the groups:

$$H = \mathbb{Z}_m \subseteq K_- = \{(e^{ip\theta}, e^{iq\theta})\},\ K_+ = \Delta S^3 \cdot H \subseteq G = S^3 \times S^3$$

where ΔS^3 is the diagonal embedding. The integers p and q are relatively prime, and either $m = 1$ (in which case H is the trivial group, and there are no restrictions on the parameters), or $m = 2$ (in which case $H = \langle(1,-1)\rangle \cong \mathbb{Z}_2$ and p is required to be even). This family naturally splits into two subfamilies, depending on the value of m. In both cases, the non-principal orbit G/K_- is homeomorphic to $S^3 \times S^2$; the difference lies in the other non-principal orbit G/K_+, and the principal orbit G/H.

Case 1. First, suppose $m = 1$. Then $G/K_+ = S^3 \times S^3/\Delta S^3$ is homeomorphic to S^3 under the map sending the coset $[(q_1, q_2)]$ to $q_1 {q_2}^{-1}$. The principal orbit is $G/H = S^3 \times S^3$. For a member $O := O_{(p,q:1)}$ of this subfamily, recall that $H^0(O) \cong H^7(O) \cong \mathbb{Z}$. Using Sequence 1 and Lemma 2.1 (with $t = 2$ and $\kappa = 4$), one easily sees that the cohomology groups are:

$$H^k(O) \cong \begin{cases} \mathbb{Z} & k = 0,2,5,7 \\ \ker\pi^* & k = 3 \\ \mathbb{Z}_r & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

where again $r = |\det(\pi^*)|$ for π^* the homomorphism from the rank two free abelian group $H^3(G/K_-) \oplus H^3(G/K_+)$ to the rank two free abelian group $H^3(G/H)$. In order for $H^3(O)$ to be trivial, the determinant of π^* must be non-zero.

As a preliminary step to finding this determinant, let v be a generator of $H^3(S^3)$. Fix a basis u_1, u_2 of $H^3(G/H) = H^3(S^3 \times S^3)$ which corresponds to the images of $v \otimes \mathbb{1}$ and $\mathbb{1} \otimes v$ under the Künneth isomorphism; that is, $u_i = p_i^*(v)$ where p_i is the projection of the i^{th} factor of $S^3 \times S^3$ onto S^3 $(i = 1, 2)$. Up to sign, this is the basis used in [13, Proposition 3.3] to show that $\operatorname{im} \pi_-^* = \langle(-q^2, p^2)\rangle$. We now find $\operatorname{im} \pi_+^* \leq H^3(S^3 \times S^3)$ with respect to the basis u_1, u_2.

Let $S^3 \overset{\Delta}{\hookrightarrow} S^3 \times S^3 \xrightarrow{\pi_+} G/K_+ \approx (S^3 \times S^3)/\Delta S^3 \approx S^3$ be the principal S^3-bundle with fiber inclusion Δ the diagonal embedding of S^3 in $S^3 \times S^3$. The composition $\pi_+ \circ \Delta$ is constant, and so is a degree zero map; the induced homomorphism $\Delta^* \circ \pi_+^*$ from $H^3(S^3)$ to itself is the trivial homomorphism. Therefore, the image of π_+^* is contained in the kernel of Δ^*. If $\sigma \in C_3(S^3)$ is a singular 3-chain, then for $i = 1, 2$:

$$\Delta^*(u_i)(\sigma) = u_i(\Delta(\sigma)) = p_i^*(v)((\sigma, \sigma)) = v(\sigma).$$

So the kernel of Δ^* is the subgroup of $H^3(S^3 \times S^3)$ generated by $u_1 - u_2$, and there is an integer n such that $\operatorname{im} \pi_+^* = \langle n(u_1 - u_2)\rangle$.

Next, consider the Serre spectral sequence (E, d) of the Borel fibration $S^3 \times S^3 \xrightarrow{\pi_+} G/K_+ \xrightarrow{\rho} \mathbb{H}P^\infty$ (here, ρ is the classifying map of the previous S^3-bundle). The differential $E_4^{0,3} \cong H^3(S^3 \times S^3) \xrightarrow{d_4} E_4^{4,0} \cong H^4(\mathbb{H}P^\infty)$ can be identified with the transgression ([16, Theorem 6.83]). By examining the definition of the transgression (as in [16, p.186]), we see in this particular instance that $\ker d_4 = \operatorname{im} \pi_+^* = \langle n(u_1 - u_2)\rangle$. Based on the convergence of the spectral sequence to $H^*(G/K_+) \cong H^*(S^3)$, we observe that $H^3(S^3 \times S^3)/\ker d_4$ must be isomorphic to $H^4(\mathbb{H}P^\infty) \cong \mathbb{Z}$. Using the basis $u_1, u_1 - u_2$ for $H^3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$, we conclude that $|n| = 1$; so the image of π_+^* in $H^3(S^3 \times S^3)$ with respect to the basis u_1, u_2 is the subgroup $\langle(1, -1)\rangle$.

From the above, the absolute value of the determinant of $\pi^* = \pi^*_- - \pi^*_+$ is $|p^2 - q^2|$. Since neither p nor q may be zero, $|p^2 - q^2| = |(p+q)(p-q)| \neq 1$, so $H^4(O)$ is non-trivial. Since p and q are relatively prime, $\det(\pi^*)$ can equal zero only if $|p| = |q| = 1$. So as long as $|p|$ and $|q|$ are not both equal to one, $H^3(O)$ is trivial and $H^4(O)$ is a non-trivial finite cyclic group. If either p or q is even, the order of $H^4(O)$ is odd.

Case 2. Let $m = 2$. In this case, $G/K_+ = (S^3 \times S^3)/(\Delta S^3 \cdot \langle(1,-1)\rangle)$ is homeomorphic to $\mathbb{R}P^3$ under the map sending the coset $[(q_1, q_2)]$ to the coset $[q_1 {q_2}^{-1}]$. The principal orbit $G/H = S^3 \times S^3/\langle(1,-1)\rangle$ is homeomorphic to $S^3 \times \mathbb{R}P^3$ under the map $[q_1, q_2] \mapsto (q_1, [q_2])$. Once again, Sequence 1 and Lemma 2.1 (with $t = 2$ and $\kappa = 4$) are sufficient tools for determining the cohomology groups of a member $O := O_{(p,q:2)}$ of this subfamily. As in the previous case, they are:

$$H^k(O) \cong \begin{cases} \mathbb{Z} & k = 0,2,5,7 \\ \ker\pi^* & k = 3 \\ \mathbb{Z}_r & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

for r the absolute value of the determinant of the homomorphism π^* from the rank two free abelian group $H^3(G/K_-)\oplus H^3(G/K_+)$ to the rank two free abelian group $H^3(G/H)$, and $H^3(O) \cong \ker\pi^*$ is trivial when the determinant of π^* is not zero.

To calculate $|\det(\pi^*)|$, we refer again to Diagram 4. As before, μ_- is the identity map. Now, however, μ_+ is the projection of the universal cover of $\mathbb{R}P^3$ by S^3, which has covering degree two; so $|\det(\mu^*)| = 2$. The composition $S^3 \times S^3 \xrightarrow{\eta} G/H \xrightarrow{\approx} S^3 \times \mathbb{R}P^3$ is the universal cover, and an argument analogous to the first argument involving Diagram 5 (used for members of the family $\mathcal{L}$ with p_+ odd) shows that $|\det(\eta^*)| = 2$.

The absolute value of the determinant of τ^* has already been computed; the homomorphism π^* that determined the order of the fourth cohomology group in the previous subfamily $\{O_{(p,q:1)}\}$ is the same as the current homomorphism τ^*.

Thus, the absolute value of the determinant of the current homomorphism π^* is $|\det(\pi^*)| = |\det(\eta^*)^{-1}||\det(\tau^*)||\det(\mu^*)| = |p^2 - q^2|$. Recall that p is even for members of this subfamily, so $H^4(O)$ is finite cyclic of odd order $r = |p^2 - q^2|$ and $H^3(O) = 0$.

4. Cohomology rings.

In this section, we show that if a manifold M is a member of the subfamily of $\mathcal{L}$ where p_+ is odd and $p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0$, or any member of the family $\mathcal{N}$, or a member of the subfamily of $\mathcal{O}$ with $|p|$ and $|q|$ not both equal to one, then the cohomology ring $H^*(M)$ is generated by classes $x \in H^2(M)$ and $y \in H^5(M)$. This will complete the proof of Theorem 1.1. Corollary 1.3 is immediate. Corollary 1.4 follows from the analysis (carried out in Section 3) of the order of the fourth cohomology groups, as expressed in terms of the parameters of the principal orbits. Using the same parametrized expressions for the orders of the fourth cohomology groups, one sees that infinitely many orders are realizable, which verifies Corollary 1.5

This section also presents an almost complete description of the cohomology ring structure of the remaining manifolds, members of the subfamily of $\mathcal{L}$ with p_+ is even. We show for any such manifold M, if classes x and y generate $H^2(M)$ and the free part of $H^5(M)$ respectively, then the class x^2 generates $H^4(M)$ and xy generates $H^7(M)$. This will complete the proof of Theorem 1.2.

All manifolds considered in this section are simply connected, and hence orientable. They also have non-principal orbits G/K_- which are closed, orientable submanifolds of codimension two. So the normal disk bundles over G/K_- are orientable bundles with fiber D^2. Thus, we have at our disposal Sequence 2 and (provided the conditions are met) Lemma 2.2, setting $t = 2$ in both. In the following, we will assume that the class x generates $H^2(M)$, the class y generates the free part of $H^5(M)$, and $\mathbb{1}_\pm$ is the multiplicative unit of $H^*(G/K_\pm)$.

4.1. Cohomology rings of the family $\mathcal{L}$.

Case 1. Let $L := L_{(p_-,q_-),(p_+,q_+)}$ be a member of the subfamily of $\mathcal{L}$ for which p_+ is odd and $p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0$. In this case, Lemma 2.2 cannot be called on to show that the square of the generator x of $H^2(L)$ generates $H^4(L)$, as Condition 3 fails. Fortunately, there is an alternate method of showing x^2 generates $H^4(L)$.

Setting $t = 2$, $k = 2$ and $B_\pm = G/K_\pm$ in Sequence 2 for the pair $(L, G/K_+)$ gives a short exact sequence:

$$0 \to H^0(G/K_-) \cong \mathbb{Z} \xrightarrow{J} H^2(L) \cong \mathbb{Z} \cdot x \xrightarrow{i_+^*} H^2(G/K_+) \cong \mathbb{Z}_2 \to 0$$

and we see that $J(\mathbb{1}_-) = \pm 2x$. Setting $k = 4$ in Sequence 2, exactness together with the triviality of $H^4(G/K_+)$ implies $J(\gamma)$ generates $H^4(L)$.

Recalling that J is an $H^*(L)$-module homomorphism:

$$2J(\gamma) = J(2\gamma) = J(i_-^*(x)) = J(\mathbb{1}_-) \smile x = \pm 2x^2.$$

Since $J(\gamma)$ generates the finite cyclic group $H^4(L)$, the subgroup generated by $2x^2 = \pm 2J(\gamma)$ is an index two subgroup. We show that x^2 is not an element of $\langle 2x^2 \rangle$, from which it follows x^2 generates $H^4(L)$. Because this argument will require both integral and $\mathbb{Z}_2$ cohomology, we temporarily resort to explicitly indicating coefficients.

The short exact sequence of abelian groups:

$$0 \to \mathbb{Z} \xrightarrow{h} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_2 \to 0$$

where h is multiplication by two and g is the natural projection, gives rise to the long exact cohomology sequence:

$$\cdots \to H^k(L;\mathbb{Z}) \xrightarrow{h_\#} H^k(L;\mathbb{Z}) \xrightarrow{g_\#} H^k(L;\mathbb{Z}_2) \xrightarrow{\beta} H^{k+1}(L;\mathbb{Z}) \to \cdots$$

where β is the Bockstein operator, and $h_\#$ and $g_\#$ are coefficient homomorphisms. Because $J(\gamma)$ generates $H^4(L;\mathbb{Z})$, and by definition $h_\#(J(\gamma)) = 2J(\gamma)$, exactness of the sequence implies $\langle 2x^2 \rangle = \langle 2J(\gamma) \rangle$ is the kernel of $g_\#$. Hence, if $g_\#(x^2)$ can be shown to be non-trivial in $H^4(L;\mathbb{Z}_2)$, it will follow that x^2 is not in $\langle 2x^2 \rangle$.

Since $H^3(L;\mathbb{Z})$ is trivial, exactness implies that the homomorphism $g_\#$ from $H^2(L;\mathbb{Z})$ to $H^2(L;\mathbb{Z}_2)$ is surjective. Thus $g_\#$ sends the generator x of $H^2(L;\mathbb{Z})$

to the generator w of $H^2(L;\mathbb{Z}_2)$. Checking the definitions of the induced homomorphism $g_\#$ and the cup product reveals that $g_\#(x^2) = w^2$.

The $\mathbb{Z}_2$-cohomology of L and the non-principal orbits $G/K_\pm$ are:

$$H^k(L;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0,2,3,4,5,7 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(G/K_-;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0,2,3,5 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(G/K_+;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0,1,2,3,4,5 \\ 0 & \text{otherwise.} \end{cases}$$

Applying Sequence 2 with $\mathbb{Z}_2$-coefficients to the pair $(L, G/K_-)$ (recall that orientability of G/K_+ is not required for $\mathbb{Z}_2$ coefficients) reveals that J is an isomorphism from $H^0(G/K_+;\mathbb{Z}_2)$ to $H^2(L;\mathbb{Z}_2)$. Under this isomorphism, if $\mathbb{1}$ is the unit of the cohomology ring $H^*(G/K_+;\mathbb{Z}_2)$, then $J(\mathbb{1}) = w$. Applying Sequence 2 to the pair $(L, G/K_+)$ shows the homomorphism i_+^* from $H^2(L;\mathbb{Z}_2)$ to $H^2(G/K_+;\mathbb{Z}_2)$ to be an isomorphism, so $i_+^*(w)$ generates $H^2(G/K_+;\mathbb{Z}_2)$. Returning to the sequence of the pair $(L, G/K_-)$, we see that $H^2(G/K_+;\mathbb{Z}_2)$ is isomorphic to $H^4(L;\mathbb{Z}_2)$ under J; hence, $J(i_+^*(w))$ generates $H^4(L;\mathbb{Z}_2)$. Since J is an $H^*(L;\mathbb{Z}_2)$-module homomorphism:

$$J(i_+^*(w)) = J(\mathbb{1} \smile i_+^*(w)) = J(\mathbb{1}) \smile w = w^2.$$

Thus, $w^2 = g_\#(x^2)$ generates $H^4(L;\mathbb{Z}_2)$. In particular, $g_\#(x^2)$ is non-trivial, which is what we needed to show in order to conclude that x^2 generates $H^4(L;\mathbb{Z})$. As $\mathbb{Z}_2$ coefficients will no longer be needed, we return to the convention of assuming integral cohomology and no longer specify coefficients.

We now show that all of the conditions of Lemma 2.2 hold when $\kappa = 7$, from which it follows that the class xy generates $H^7(L)$. Recall that $t = 2$, and observe that all conditions on the cohomology groups are met. Since $H^1(G/K_-)$ is trivial, i_+^* from $H^2(L)$ to $H^2(G/K_+)$ is a surjection.

We check the remaining three conditions. Since $H^7(G/K_+)$ is trivial, i_+^* from $H^7(L)$ to $H^7(G/K_+)$ is the zero homomorphism, and Condition 1 holds. Condition 2 also holds, with $H^5(G/K_-) \cong \mathbb{Z} \cdot \nu$. The group $H^2(G/K_+)$ is finite cyclic of order two, and $H^7(L)$ is infinite cyclic; so to verify Condition 3, we need to show that $i_+^*(y) = \pm 2\nu$.

Take $k = 5$ in Diagram 3. Triviality of $H^4(G/K_-)$ implies that $H^5(L, G/K_-)$ injects into the infinite cyclic group $H^5(L)$, while triviality of $H^4(G/H)$ implies that $H^5(D(G/K_+), G/H)$ injects into the finite cyclic group $H^5(G/K_+)$. Because the groups $H^5(L, G/K_-)$ and $H^5(D(G/K_+), G/H)$ are isomorphic, the existence of these two simultaneous injections implies that $H^5(L, G/K_-)$ and $H^5(D(G/K_+), G/H)$ must both be trivial. This yields two short exact sequences:

$$0 \to H^5(L) \cong \mathbb{Z} \cdot y \xrightarrow{i_-^*} H^5(G/K_-) \cong \mathbb{Z} \cdot \nu \xrightarrow{\delta_-} H^5(L, G/K_-) \to 0$$
$$0 \to H^5(G/K_+) \cong \mathbb{Z}_2 \xrightarrow{i^*} H^5(G/H) \cong \mathbb{Z}_4 \xrightarrow{\delta} H^5(D(G/K_+), G/H) \to 0$$

where the groups $H^5(L, G/K_-)$ and $H^5(G/K_+, G/H)$ are isomorphic. From the second sequence, we conclude $H^6(D(G/K_+), G/H) \cong \mathbb{Z}_2$. It is then apparent by the first sequence that i_-^* is multiplication by two; so $i_-^*(y) = \pm 2\nu$, Condition 3 holds, and by Lemma 2.2, xy generates $H^7(L)$.

Case 2. Suppose $L := L_{(p_-,q_-),(p_+,q_+)}$ is a member of the subfamily of $\mathcal{L}$ with p_+ even. Let x generate $H^2(L) \cong \mathbb{Z}$ and y the free part of $H^5(L) \cong \mathbb{Z} \oplus \mathbb{Z}_2$; we show that x^2 generates $H^4(L)$ and xy generates $H^7(L)$. Whether or not the classes x and y, together with a class ξ generating $H^3(L) \cong \mathbb{Z}_2$, form a complete set of generators for the ring $H^*(L)$ is unknown at this time.

Setting $t = 2$, we confirm that the conditions of Lemma 2.2 hold when $\kappa = 4, 7$. The conditions on $H^2(L)$ and $H^2(G/K_+)$, as well as Conditions 1 and 2, are easily checked. Sequence 2 can be used to verify that the inclusion-induced homomorphism from $H^2(L) \cong \mathbb{Z}$ to $H^2(G/K_+) \cong \mathbb{Z}_4$ is a surjection, and those from $H^\kappa(L)$ to $H^\kappa(G/K_+)$ (for $\kappa = 4, 7$) are multiplication by zero. It remains to check Condition 3.

When $\kappa = 4$, $H^4(L)$ is finite cyclic. We show $i_-^*(x)$ generates $H^2(G/K_-)$

when x generates $H^2(L)$. Consider Diagram 3. The second relative cohomology groups, which are isomorphic, inject into both a free group and a finite group and so must be trivial. Because $H^3(L) \cong \mathbb{Z}_2$ is in the kernel of i_-^*, the third relative group $H^3(L, G/K_-)$ surjects onto $H^3(L)$. By exactness of the top row, $H^3(L, G/K_-)/\operatorname{im}\delta$ is isomorphic to $\mathbb{Z}_2$ and $\operatorname{im}\delta$ is finite cyclic. We conclude that $H^3(L, G/K_-)$ is a non-trivial finite group.

Triviality of $H^2(D(G/K_+), G/H)$ implies that $H^2(G/K_+) \cong \mathbb{Z}_4$ injects into $H^2(G/H) \cong \mathbb{Z}_4$; so $H^2(G/K_+)$ and $H^2(G/H)$ are isomorphic. It follows from the isomorphism of the third relative cohomology groups, together with exactness in the bottom row, that $H^3(L, G/K_-)$ injects into $H^3(G/K_+) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. So $H^3(L, G/K_-)$ is isomorphic to $\mathbb{Z}_2$, and the surjection of $H^3(L, G/K_-)$ onto $H^3(L)$ is an isomorphism. Then, by exactness of the top row, the inclusion-induced homomorphism i_-^* from $H^2(L) \cong \mathbb{Z} \cdot x$ to $H^2(G/K_-)$ must also be an isomorphism, and $i_-^*(x)$ generates $H^2(G/K_-)$. This satisfies Condition 3 in the case $\kappa = 4$.

If $\kappa = 7$, $H^7(L)$ is infinite cyclic. We show that, for y a generator of the free part of $H^5(L)$, $i_-^*(y)$ is four times a generator of $H^5(G/K_-) \cong \mathbb{Z}$. Again turning to Diagram 3, we see that the fifth relative cohomology groups, which are isomorphic, inject into both $H^5(L) \cong \mathbb{Z}\cdot y \oplus \mathbb{Z}_2$ and $H^5(G/K_+) \cong \mathbb{Z}_2$. Hence, they are either trivial or cyclic of order two. Since the $\mathbb{Z}_2$ summand of $H^5(L)$ is in the kernel of the homomorphism i_-^* from $H^5(L)$ to $H^5(G/K_-)$, we conclude that the fifth relative cohomology groups are isomorphic to $\mathbb{Z}_2$. Then exactness of the bottom row together with the isomorphism of the relative groups gives an isomorphism between $H^6(L, G/K_-)$ and $H^5(G/H) \cong \mathbb{Z}_4$. Restricting i_-^* to the free part of $H^5(L)$ gives rise to a short exact sequence:

$$0 \to \mathbb{Z} \cdot y \xrightarrow{i_-^*|_{\mathbb{Z}}} H^5(G/K_-) \cong \mathbb{Z} \xrightarrow{\delta_-} \mathbb{Z}_4 \to 0.$$

Hence, $i_-^*(y)$ is four times a generator of $H^5(G/K_-)$, satisfying Condition 3 in the case $\kappa = 7$, and by Lemma 2.2 it follows that x^2 generates $H^4(L)$ and xy generates $H^7(L)$.

This completes the proof of Theorem 1.2. $\square$

4.2. Cohomology rings of the family $\mathcal{N}$.

We continue with the proof of Theorem 1.1.

Let $N := N_{(p_-,q_-),(p_+,q_+)}$ be a member of the family $\mathcal{N}$. The cohomology groups of the orbits (as computed in [11, Lemma 13.6]) are:

$$H^k(G/K_-) \cong \begin{cases} \mathbb{Z} & k = 0,3,5 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & k = 2 \\ \mathbb{Z}_2 & k = 4 \\ 0 & \text{otherwise} \end{cases} \qquad H^k(G/K_+) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}_4 & k = 2 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & k = 3 \\ \mathbb{Z}_2 & k = 5 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(G/H) \cong \begin{cases} \mathbb{Z} & k = 0,6 \\ \mathbb{Z}_2\oplus\mathbb{Z}_4 & k = 2,5 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2 & k = 3 \\ \mathbb{Z}_2 & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

The classes x and y respectively generate the infinite cyclic groups $H^2(N)$ and $H^5(N)$. To show that x^2 generates $H^4(N)$ and xy generates $H^7(N)$, we turn to Lemma 2.2 (recall that $t = 2$). For $\kappa = 4$ and 7, all of the conditions on the cohomology groups are satisfied, including Condition 2. In particular, $H^2(G/K_+)$ is finite cyclic of order $n = 4$. Taking $k = 2$ in Sequence 2, one sees that the inclusion-induced homomorphism from $H^2(N)$ to $H^2(G/K_+)$ is a surjection, and also that the inclusion-induced homomorphisms from $H^\kappa(N)$ to $H^\kappa(G/K_+)$, $\kappa = 4$ and 7, are the zero homomorphisms (Condition 1). It remains only to check that the requirements of Condition 3 are satisfied.

For $\kappa = 4$, the group $H^4(N)$ is finite cyclic. Suppose the image of x under the inclusion-induced homomorphism i_-^* is the element (s,β) in $H^2(G/K_-) \cong \mathbb{Z}\oplus\mathbb{Z}_2$. In order for Condition 3 to hold, s must be relatively prime to the order of $H^4(N)$. We claim this is true; that, in fact, $|s| = 1$. To see this, consider Diagram 3. Setting $k = 2$, we see that the second relative cohomology groups, which are isomorphic, inject into both the infinite cyclic group $H^2(N)$ and the finite cyclic group $H^2(G/K_+)$; so they must be trivial. Because $H^3(N)$ is trivial, the homomorphism δ_- from $H^2(G/K_-)$ to $H^3(N, G/K_-)$ is a surjection. Commutativity of the diagram together with the isomorphism of the third relative groups forces the homomorphism δ from $H^2(G/H)$ to $H^3(D(G/K_+), G/H)$ to be surjective as well. This gives two short exact sequences:

$$0 \to H^2(N) \cong \mathbb{Z} \cdot x \xrightarrow{i_-^*} H^2(G/K_-) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\delta_-} H^3(N, G/K_-) \to 0$$
$$0 \to H^2(G/K_+) \cong \mathbb{Z}_4 \xrightarrow{i^*} H^2(G/H) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \xrightarrow{\delta} H^3(D(G/K_+), G/H) \to 0$$

where the relative cohomology groups are isomorphic. From the second sequence we see that the order of the relative groups is the order of $H^2(G/H)$ divided by the order of $H^2(G/K_+)$; hence, the relative groups are isomorphic to $\mathbb{Z}_2$.

Consider the first sequence. Because i_-^* is injective and $i_-^*(x) = (s, \beta)$, s cannot be zero. By exactness, $H^3(N, G/K_-) \cong \mathbb{Z}_2$ is isomorphic to $H^2(G/K_-)/\operatorname{im} i_-^*$. If $\beta = [0]$, the group $H^2(G/K_-)/\langle(s, \beta)\rangle$ is clearly isomorphic to $\mathbb{Z}_s \oplus \mathbb{Z}_2$. If instead $\beta = [1]$, the surjective homomorphism from $H^2(G/K_-) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ to $\mathbb{Z}_{2s}$, defined by $(1, [0]) \mapsto [1]$ and $(0, [1]) \mapsto [s]$, has kernel $\langle(s, [1])\rangle$; hence, $H^2(G/K_-)/\langle(s, [1])\rangle$ is isomorphic to $\mathbb{Z}_{2s}$. In both cases, $H^2(G/K_-)/\operatorname{im} i_-^*$ is a finite group with $2|s|$ elements. Because we know $H^2(G/K_-)/\operatorname{im} i_-^*$ is isomorphic to $\mathbb{Z}_2$, we conclude $|s| = 1$. Thus, s is relatively prime to the order of $H^4(N)$, Condition 3 is satisfied and Lemma 2.2 holds for $\kappa = 4$. We have shown x^2 generates $H^4(N)$.

We now show that Condition 3 of Lemma 2.2 holds for $\kappa = 7$, from which it follows xy generates $H^7(N)$. Because $H^7(N)$ is infinite cyclic and $H^2(G/K_+)$ is finite cyclic of order $n = 4$, Condition 3 requires the image of the generator y of $H^5(N)$ under i_-^* to be (up to sign) four times a generator of the infinite cyclic group $H^5(G/K_-)$.

To show this is true, set $k = 5$ in Diagram 3. The sixth relative cohomology groups are isomorphic, and by exactness of the bottom row are isomorphic to the quotient of $H^5(G/H)$ by $i^*(H^5(G/K_+))$. So the orders of the sixth relative groups are equal to the order of $H^5(G/H) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ divided by the order of

$i^*(H^5(G/K_+))$. Since $H^5(G/K_+) \cong \mathbb{Z}_2$, this is either four or eight. Observe that these relative groups cannot contain elements of order eight, since they are isomorphic to a quotient of $\mathbb{Z}_2 \oplus \mathbb{Z}_4$; therefore, if they are eight element groups, they cannot be cyclic. But the infinite cyclic group $H^5(G/K_-)$ surjects onto the relative cohomology groups, so we conclude they are cyclic of order four. It then follows from exactness of the top row that the homomorphism i_-^* from $H^5(N)$ to $H^5(G/K_-)$ is multiplication by four. Hence, by Lemma 2.2, xy generates $H^7(N)$.

4.3. Cohomology rings of the family $\mathcal{O}$.

Let $O := O_{(p,q:m)}$ be a member of this family with $|p|$ and $|q|$ not both equal to one. The cases $m = 1$ and $m = 2$ need to be considered separately, due to differences in the orbits G/H and G/K_+. However, calculations are greatly simplified by the fact that both non-principal orbits are orientable. This means Sequence 2 holds for both of the pairs $(O, G/K_\pm)$. Also, despite the different orbits, arguments for each of the cases $m = 1, 2$ are similar; we sketch the general method.

For $\kappa = 4, 7$, all conditions of Lemma 2.2 applying to the cohomology groups (including Condition 2) are met. As before, $t = 2$. Sequence 2 applied to the pair $(O, G/K_+)$ can be used to show that $H^2(O)$ surjects onto $H^2(G/K_+)$. This same sequence can be used to show that the homomorphisms i_+^* from $H^\kappa(O)$ to $H^\kappa(G/K_+)$ for $\kappa = 4$ and 7 are trivial homomorphisms; consequently Condition 1 holds. Finally, applying Sequence 2 to the pair $(O, G/K_-)$, one sees that under the homomorphism i_-^*, the image of a generator of $H^{\kappa-2}(O)$ meets the requirements of Condition 3. Thus, by Lemma 2.2, $H^*(O)$ is generated by $x \in H^2(O)$ and $y \in H^5(O)$.

This completes the proof of Theorem 1.1. □

We conclude with a table summarizing the cohomological data for compact, simply connected, seven dimensional, primitive cohomogeneity one manifolds:

Family	$\mathcal{L} := \{L_{(p_-,q_-),(p_+,q_+)}\}$
Parameter restrictions	p_+ odd & $p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0$
Cohomology groups	$H^k(L;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0,2,5,7 \\ \mathbb{Z}_r,\ r \neq 1,0 & k = 4 \\ 0 & \text{otherwise} \end{cases}$
Order of $H^4(L;\mathbb{Z})$	$r = \frac{1}{4}\lvert p_+^2 q_-^2 - p_-^2 q_+^2 \rvert$
Ring generators	$x \in H^2(L;\mathbb{Z})$ and $y \in H^5(L;\mathbb{Z})$
Notes	r is always even. Kreck-Stolz invariants exits.

Family	$\mathcal{L} := \{L_{(p_-,q_-),(p_+,q_+)}\}$
Parameter restrictions	p_+ even
Cohomology groups	$H^k(L;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0,2,7 \\ \mathbb{Z}_2 & k = 3 \\ \mathbb{Z}_r,\ r \neq 0,1 & k = 4 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & k = 5 \\ 0 & \text{otherwise} \end{cases}$
Order of $H^4(L;\mathbb{Z})$	$r = \lvert p_+^2 q_-^2 - p_-^2 q_+^2 \rvert$
Ring generators (partial list)	Let $H^2(L;\mathbb{Z}) = \mathbb{Z} \cdot x,\ H^5(L;\mathbb{Z}) = \mathbb{Z} \cdot y \oplus \mathbb{Z}_2$; then x^2 generates $H^4(L;\mathbb{Z})$, and xy generates $H^7(L;\mathbb{Z})$.
Notes	r is always odd.

Family	$\mathcal{M} := \{M_{(p_-,q_-),(p_+,q_+)}\}$
Restrictions	$p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0$
Cohomology groups	$H^k(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, 7 \\ \mathbb{Z}_r,\ r \neq 0, 1 & k = 4 \\ 0 & \text{otherwise} \end{cases}$
Order of $H^4(M;\mathbb{Z})$	$r = \frac{1}{8}\lvert p_+^2 q_-^2 - p_-^2 q_+^2\rvert$
Ring generators	$y \in H^4(M;\mathbb{Z})$ and $z \in H^7(M;\mathbb{Z})$
Notes	Computed in [11]; same cohomology ring as an S^3-bundle over S^4.

Family	$\mathcal{N} := \{N_{(p_-,q_-),(p_+,q_+)}\}$
Parameter restrictions	None.
Cohomology groups	$H^k(N;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, 2, 5, 7 \\ \mathbb{Z}_r,\ r \neq 0, 1 & k = 4 \\ 0 & \text{otherwise} \end{cases}$
Order of $H^4(N;\mathbb{Z})$	$r = \lvert p_+^2 q_-^2 - p_-^2 q_+^2\rvert$
Ring generators	$x \in H^2(N;\mathbb{Z})$ and $y \in H^5(N;\mathbb{Z})$
Notes	Groups computed in [11]. r is always odd. Kreck-Stolz invariants exist.

Family	$\mathcal{O} := \{O_{(p,q:m)}\}$
Parameter restrictions	either $\lvert p\rvert \neq 1$ or $\lvert q\rvert \neq 1$
Cohomology groups	$H^k(O;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, 2, 5, 7 \\ \mathbb{Z}_r,\ r \neq 0, 1 & k = 4 \\ 0 & \text{otherwise} \end{cases}$
Order of $H^4(O;\mathbb{Z})$	$r = \lvert p^2 - q^2\rvert$
Ring generators	$x \in H^2(O;\mathbb{Z})$ and $y \in H^5(O;\mathbb{Z})$
Notes	r is odd whenever either p or q is even. Kreck-Stolz invariants exist.

Table 2: Cohomological data for compact, simply connected, seven dimensional, primitive cohomogeneity one manifolds.

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