NON-NEGATIVELY CURVED 6-MANIFOLDS WITH ALMOST MAXIMAL SYMMETRY RANK

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Abstract. We classify closed, simply-connected, non-negatively curved 6-manifolds of almost maximal symmetry rank up to equivariant diffeomorphism.

1. Introduction

For the class of closed, simply-connected Riemannian manifolds there are no known obstructions that allow us to distinguish between positive and non-negative sectional curvature, in spite of the fact that the number of known examples of manifolds of non-negative sectional curvature is vastly larger than those known to admit a metric of positive sectional curvature.

The introduction of symmetries, however, allows us to distinguish between such classes. An important first case to understand is that of maximal symmetry rank, where $\mathrm{symrk}(M^n) = \mathrm{rk}(\mathrm{Isom}(M^n))$. For manifolds of strictly positive sectional curvature, a classification up to equivariant diffeomorphism was obtained by Grove and Searle [12]. They showed that for such manifolds, the maximal symmetry rank is equal to $\lfloor (n+1)/2 \rfloor$. For closed, simply-connected manifolds of non-negative sectional curvature, the maximal symmetry rank is conjectured to be $\lfloor 2n/3 \rfloor$ (see Galaz-García and Searle [9] and Escher and Searle [6]). A classification for the latter has been obtained, but only in dimensions less than or equal to nine (see [9] and Galaz-García and Kerin [8], for dimensions less than or equal to 6 and [6] for dimensions 7 through 9) and the upper bound for the symmetry rank has been verified for dimensions less than or equal to 12 (see [9] and [6]).

A natural next step is the case of almost maximal symmetry rank. In positive curvature, a homeomorphism classification was obtained by Rong [31], in dimension 5, and Fang and Rong [7], for dimensions greater than or equal to 8, using work of Wilking [36]. In non-negative curvature, a homeomorphism classification was obtained independently by Kleiner [17] and Searle and Yang [32], in dimension 4. This classification was later improved to equivariant diffeomorphism by Grove and Wilking [13]. A diffeomorphism classification in dimension 5 was obtained by Galaz-García and Searle [10].

In this article we consider closed, simply-connected Riemannian 6-manifolds admitting a metric of non-negative sectional curvature and an effective, isometric torus action of almost maximal symmetry rank and prove the following classification theorem.

Theorem A. *Let T^3 act isometrically and effectively on M^6, a closed, simply-connected, non-negatively curved Riemannian manifold. Then M^6 is equivariantly diffeomorphic to $S^3 \times S^3$ or a torus manifold.*

Closed, orientable manifolds of dimension $2n$ admitting a smooth T^n action with non-empty fixed point set, are called *torus manifolds*. Non-negatively curved torus manifolds were classified up to equivariant diffeomorphism by Wiemeler [35] (see Theorem 2.13). In dimension 6, they are equivariantly diffeomorphic to S^6, $\mathbb{C}P^3 = S^7/T^1$, or the quotient by (1), a free linear circle action on $S^3 \times S^4$, (2), a free linear T^2-action on $S^3 \times S^5$ or (3), a free linear T^3-action on $S^3 \times S^3 \times S^3$. In the process of classifying bi-quotients of dimension 6, De Vito [5] has given a classification of these manifolds up to diffeomorphism. It is worth noting that Kuroki [18], using torus graphs, has

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obtained an Orlik-Raymond type classification of 6-dimensional torus manifolds with vanishing odd degree cohomology without curvature restrictions.

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2. Preliminaries

In this section we will gather basic results and facts about transformation groups, the topological classification of six dimensional manifolds, cohomogeneity two torus actions and G-invariant manifolds of non-negative curvature.

2.1. Transformation Groups.

Let G be a compact Lie group acting on a smooth manifold M. We denote by $G_x = \{ g \in G : gx = x \}$ the *isotropy group* at $x \in M$ and by $G(x) = \{ gx : g \in G \} \simeq G/G_x$ the *orbit* of x. Orbits are called *principal*, *exceptional* or *singular*, depending on the relative size of their isotropy subgroups; that is, principal orbits correspond to those orbits with the smallest possible isotropy subgroup, an orbit is called exceptional when its isotropy subgroup is a finite extension of the principal isotropy subgroup and singular when its isotropy subgroup is of strictly larger dimension than that of the principal isotropy subgroup.

The *ineffective kernel* of the action is the subgroup $K = \cap_{x \in M} G_x$. We say that G acts *effectively* on M if K is trivial. The action is called *almost effective* if K is finite.

We will sometimes denote the *fixed point set* $M^G = \{ x \in M : gx = x, g \in G \}$ of the G-action by $\mathrm{Fix}(M; G)$. One measurement for the size of a transformation group $G \times M \to M$ is the dimension of its orbit space M/G, also called the *cohomogeneity* of the action. This dimension is clearly constrained by the dimension of the fixed point set M^G of G in M. In fact, $\dim(M/G) \geq \dim(M^G) + 1$ for any non-trivial action with fixed points. In light of this, the *fixed-point cohomogeneity* of an action, denoted by $\mathrm{cohomfix}(M; G)$, is defined by

$$\mathrm{cohomfix}(M; G) = \dim(M/G) - \dim(M^G) - 1 \geq 0.$$

A manifold with fixed-point cohomogeneity 0 is also called a *G-fixed point homogeneous manifold.*

2.2. Topological Classification of 6-manifolds.

Note that throughout the paper we will use the convention that all homology groups have integer coefficients, unless otherwise specified.

The topological classification of simply-connected, closed, oriented 6-manifolds has been completed in a sequence of articles by C.T.C. Wall [34], P. Jupp [15], and A. Žubr [37, 38, 39]. We will focus on the classification of closed, simply-connected, oriented 6-manifolds with torsion free homology. The classification theorem below is due to C. T. C. Wall in the case of smooth spin manifolds, [34], and in the final form due to P. Jupp [15]. We first describe the basic invariants used to classify 6-dimensional, closed, simply-connected, oriented, smooth manifolds, M, with torsion free homology [15].

Theorem 2.1. [15] *Let M be a 6-dimensional, closed, simply-connected, oriented, smooth manifold with torsion free homology. The basic invariants used to classify M are as enumerated below.*

1. $H := H^2(M)$, *a finitely generated free abelian group;*
2. $b := b_3(M) = rk_{\mathbb{Z}}(H^3(M)) \in 2\mathbb{Z}$ *since* $H^3(M)$ *admits a non-degenerate symplectic form;*
3. $F := F_M : H^2(M) \otimes H^2(M) \otimes H^2(M) \longrightarrow \mathbb{Z}$ *a symmetric trilinear form given by the cup product evaluated on the orientation class;*
4. $p := p_1(M) \in H^4(M)$, *the first Pontrjagin class;*
5. $w := w_2(M) \in H^2(M; \mathbb{Z}_2)$, *the second Stiefel-Whitney class.*

We now use Poincaré duality to identify $H^4(M)$ with $\mathrm{Hom}_{\mathbb{Z}}(H^2(M); \mathbb{Z})$ so that $p_1(M)$ can be interpreted as a linear form on $H^2(M)$ and we let $x \cdot y \cdot z$ denote $F_M(x \otimes y \otimes z)$ for $x, y, z \in H^2(M)$.

Definition 2.2. (Admissibility)

The system of invariants (H, b, w, F, p) is called admissible *if and only if for every $\omega \in H$ and $T \in H^* := Hom_{\mathbb{Z}}(H;\mathbb{Z})$ with $\rho_2(\omega) = w$ and $\rho_2(T) = 0$ where $\rho_2 : \mathbb{Z} \longrightarrow \mathbb{Z}_2$ is reduction modulo 2, the following congruence holds:*

$$\omega^3 \equiv (p + 24T)\,\omega \mod 48.$$

Definition 2.3. (Equivalence) *Two systems (H, b, w, F, p) and (H', b', w', F', p') are called* equivalent *if and only if $b = b'$ and there exists an isomorphism $\alpha : H \longrightarrow H'$ such that $\alpha(w) = w', \alpha^*(F') = F, \alpha^*(p') = p$.*

We are now ready to state the classification result:

Theorem 2.4. [15] *The assignment*

$$M \mapsto (\frac{b}{2}, H^2(M), w_2(M), F_M, p_1(M))$$

induces a 1-1 correspondence between oriented diffeomorphism classes of simply-connected, closed, oriented, 6-dimensional, smooth manifolds with torsion free homology, and equivalence classes of admissible systems of invariants.

Note that A. Žubr generalized Wall's theorem in a different direction: he proved a classification theorem for simply-connected, smooth spin manifolds with not necessarily torsion free homology [37], and then in [38, 39] also obtained Jupp's theorem and proved that algebraic isomorphisms of systems of invariants can always be realized by orientation preserving diffeomorphisms.

Observe that the first invariant $\frac{b}{2}$ is completely independent of the other invariants which implies that the following splitting theorem holds.

Corollary 2.5. [34] *Every simply-connected, closed, oriented, 6-dimensional, smooth manifold M admits a splitting $M = M_0 \sharp \frac{b}{2}(S^3 \times S^3)$ as a connected sum of a core M_0 with $b = b_3(M_0) = 0$ and $\frac{b}{2}$ copies of $S^3 \times S^3$.*

The following corollary is an immediate consequence.

Corollary 2.6. *Let M be a simply-connected, closed, oriented, 6-dimensional, smooth manifold with*

$$H_i(M^6) \cong H_i(S^3 \times S^3) \text{ for all } i.$$

Then M^6 is diffeomorphic to $S^3 \times S^3$.

2.3. **Cohomogeneity two torus actions.** In order to prove Theorem A, we will also need to understand cohomogeneity two torus actions on smooth manifolds. In [25], Orlik and Raymond obtain an equivariant classification theorem for cohomogeneity two torus actions on smooth manifolds that states that two such manifolds are equivariantly diffeomorphic if and only if their respective weighted orbit spaces are weight-preserving diffeomorphic.

The following theorem classifies the weighted orbit spaces of almost free, cohomogeneity two torus actions on smooth manifolds with S^2 as quotient space. Moreover, by Theorems 12.3 and 12.15 of Conner and Raymond [3], M^{k+2} admits a fibration by a 3-manifold, M^3, over T^{k-1} and there is an almost free T^1 action on M^3, with $M^{k+2}/T^k = M^3/T^1$. We also identify the corresponding 3-manifolds in the following theorem, as this information will be useful in what follows. One then classifies the M^{k+2} up to (weak) equivariant diffeomorphism via their weighted orbit spaces (see Theorem 1.4, the theorem in Section 2.4 and Remark 2.5 in [25] for the weighted orbit spaces and the proof of Theorem 4 in [24] and Section 12 of [3] for the homology results).

Recall that if T^k acts smoothly, effectively and almost freely on M^{k+2}, with $M^{k+2}/T^k = S^2$, then there are a finite number of isolated orbits with finite isotropy and the isotropy subgroups must be cyclic. The possible weighted orbit spaces are then determined completely by the number of isolated exceptional orbits and are described by an ordered pair $(n, \{\alpha_i\})$, where n corresponds to the number of isolated exceptional orbits and each $\alpha_j \in \{\alpha_i\}$ corresponds to the order of the corresponding isotropy subgroup.

Theorem 2.7. [25] *Let T^k act smoothly, effectively and almost freely on M^{k+2}, a smooth $(k+2)$-dimensional manifold, with $M^{k+2}/T^k = S^2$. Then the possible weighted orbits spaces and the corresponding fiber M^3 of the fibration $M^3 \hookrightarrow M^{k+2} \to T^{k-1}$ are:*

(1) $(0, \emptyset)$ *(hence $M^{k+2} = T^k \times S^2$) and M^3 is $S^2 \times S^1$;*
(2) $(1, \{\alpha_1\})$ *and M^3 is a lens space;*
(3) $(2, \{\alpha_1, \alpha_2\})$ *and M^3 is a lens space;*
(4) $(3, \{\alpha_1, \alpha_2, \alpha_3\})$*, and $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 > 1$ and M^3 is S^3/Γ, where Γ is one of D^*, T^*, O^*, or I^*;*

In Cases (1) *through* (3)*, the fundamental group is abelian and in Case* (4)*, the fundamental group is non-abelian and has elements of finite order.*

2.4. ***G*-manifolds with non-negative curvature.** We now recall some general results about G-manifolds with non-negative curvature which we will use throughout. Recall that fixed point homogeneous manifolds of positive curvature were classified in [12]. More recently Spindeler [33] proved the following theorem which characterizes non-negatively curved G-fixed point homogeneous manifolds.

Theorem 2.8. [33] *Assume that G acts fixed point homogeneously on a complete non-negatively curved Riemannian manifold M. Let F be a fixed point component of maximal dimension. Then there exists a smooth submanifold N of M, without boundary, such that M is diffeomorphic to the normal disk bundles $D(F)$ and $D(N)$ of F and N glued together along their common boundaries;*

$$M = D(F) \cup_\partial D(N).$$

Further, N is G-invariant and contains all singularities of M up to F.

Let $\mathrm{Isom}_F(M)$ be the subgroup of the isometry group of M that leaves F invariant. The following lemma will also be important:

Lemma 2.9. [33] *Let M be a non-negatively curved fixed point homogeneous G-manifold, with M, F and N as in Theorem 2.8 and $H = \mathrm{Isom}_F(M)$. Then there exists an H-equivariant diffeomorphism $b : \partial D(N) \to \partial D(F)$ and M is H-equivariantly diffeomorphic to $D(F) \cup_\partial D(N)$.*

Since fixed point homogeneous manifolds with either positive or non-negative lower curvature bounds decompose as unions of disk bundles, the following lemma from [6] will be useful.

Lemma 2.10. [6] *Let M be a manifold with $rk(H_1(M)) = k$, $k \in \mathbb{Z}^+$. If M admits a disk bundle decomposition*

$$M = D(N_1) \cup_E D(N_2),$$

where N_1, N_2 are smooth submanifolds of M and N_1 is orientable and of codimension two, then both $rk(H_1(N_1))$ and $rk(H_1(N_2))$ are less than or equal to $k+1$.

Moreover, we have the following theorem which allows us to identify the fundamental group of E in the disk bundle decomposition.

Theorem 2.11. *Let M^n be a simply-connected manifold that decomposes as the union of two disk bundles as follows:*

$$M^n = D^k(N_1) \cup_E D^l(N_2).$$

Then the following hold:

(1) *If $k = l = 2$ and $\pi_2(N_i) = 0$ for $i = 1, 2$ and $\pi_1(N_1)$ is not a finite cyclic group, then $\pi_1(E) \cong \mathbb{Z}^2$.*
(2) *If $k \geq 3$, then $\pi_1(E) \cong \pi_1(N_1)$.*

Proof. Case (1): Assume that $k = l = 2$. Then E is a circle bundle over N_1 and also over N_2, where $\pi_2(N_1) = \pi_2(N_2) = 0$. Hence we obtain the following short exact sequences from the long exact sequences in homotopy:

$$0 \longrightarrow \pi_1(S^1_j) \xrightarrow{i_j *} \pi_1(E) \xrightarrow{f_j^*} \pi_1(N_j) \longrightarrow 0\,, \text{ for } j \in \{1,2\}$$

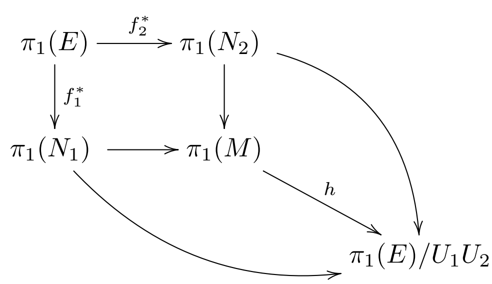
Now let $U_1 = i_1^*(\pi_1(S^1_1))$ and $U_2 = i_2^*(\pi_1(S^1_2))$. Then $\pi_1(N_1) = \pi_1(E)/U_1$ and $\pi_1(N_2) = \pi_1(E)/U_2$, so we get the following commutative diagram:

$$(*) \qquad \begin{CD} \pi_1(E) @>{f_2^*}>> \pi_1(N_2) \\ @V{f_1^*}VV @VVV \\ \pi_1(N_1) @>>> \pi_1(E)/U_1U_2 \end{CD}$$

Here the lower map is given by

$$\pi_1(N_1) = \pi_1(E)/U_1 \longrightarrow \pi_1(E)/U_1U_2\,,$$

and the same is true for $\pi_1(N_2)$. Now by Seifert Van-Kampen (universal property), there exists a morphism $h : \pi_1(M) \longrightarrow \pi_1(E)/U_1U_2$ making the following diagram commute:



Since all the maps in $(*)$ are surjective, h must be surjective. But since $\pi_1(M) = 0$, this implies that $\pi_1(E) \cong U_1U_2$. Note that both U_1 and U_2 are normal in $\pi_1(E)$. If in addition $U_1 \cap U_2 = \{1\}$, then $\pi_1(E) \cong U_1 \times U_2 \cong \mathbb{Z}^2$ and the theorem follows. If $U_1 \cap U_2 \neq \{1\}$, then $\pi_1(N_1) \cong U_1U_2/U_1 \cong U_2/U_1 \cap U_2$. But $U_1 \cap U_2$ is a normal subgroup of $U_2 \cong \mathbb{Z}$, hence $U_1 \cap U_2 \cong n\,\mathbb{Z}$ for some $n \in \mathbb{Z}$. It follows that $\pi_1(N_1) \cong U_2/U_1 \cup U_2 \cong \mathbb{Z}/n\mathbb{Z}$ which is a contradiction to the hypothesis that $\pi_1(N_1)$ is not finite cyclic. Hence $U_1 \cap U_2 = \{1\}$ and $\pi_1(E) \cong U_1 \times U_2 \cong \mathbb{Z}^2$.

Case (2): Assume now that $k \geq 3$. Then E is a S^{k-1} bundle over N_1 and hence by the long exact sequence in homotopy $\pi_1(E) \cong \pi_1(N_1)$.

□

An important subclass of manifolds admitting an effective torus action are the so-called *torus manifolds.*

Definition 2.12 (**Torus Manifold**)**.** *A* torus manifold *M is a $2n$-dimensional closed, connected, orientable, smooth manifold with an effective smooth action of an n-dimensional torus T such that $M^T \neq \emptyset$.*

Note that torus manifolds satisfy the following properties. The action of T^n on M^{2n} is an example of a *maximal* torus action, where we define a *maximal T^k action* to be one where the dimension of the smallest orbit is $2k - n$ (see Ishida [14]).

The following important theorem from [35] gives a classification up to equivariant diffeomorphism of non-negatively curved torus manifolds.

Theorem 2.13. [35] *Let M be a simply-connected, non-negatively curved torus manifold. Then M is equivariantly diffeomorphic to a quotient of a free linear torus action of*

$$N = \prod_{i<r} S^{2n_i} \times \prod_{i \geq r} S^{2n_i - 1},\ n_i \geq 2. \tag{2.3}$$

We now recall Theorem 5.1 from [6], which will be important for the proof of Theorem A.

Theorem 2.14. [6] *Let T^k act isometrically, effectively and almost maximally on M^n, a simply-connected, closed, non-negatively curved Riemannian n-manifold with $k \geq \lfloor (n+1)/2 \rfloor$. Then the action is maximal.*

3. Proof of Theorem A

In this section we present the proof of Theorem A. We first recall the following lemma from [10].

Lemma 3.1. [10] *Let T^n act on M^{n+3}, a closed, simply-connected smooth manifold. Then T^n cannot act freely or almost freely; that is, some circle subgroup has non-trivial fixed point set.*

Note that by Lemma 3.1, a T^3-action on a closed, simply-connected M^6 must have circle isotropy. Therefore, we may break the proof of Theorem A into three cases, depending on the rank of the largest isotropy subgroup, which will be either 1, 2 or 3. Observe that Theorem 2.14 tells us that if the action has either T^2 or T^3 isotropy, then in fact it has T^3 isotropy and must therefore be a torus manifold. Theorem 2.13 then gives us an equivariant diffeomorphism classification of these manifolds. Thus, we have proven Part (1) of the following theorem.

Theorem 3.2. *Let M^6 be a closed, simply-connected, non-negatively curved Riemannian 6-manifold admitting an isometric, effective T^3-action. Then the action has singular isotropy of rank 1, 2 or 3 and the following hold.*

(1) *If the rank of the largest singular isotropy subgroup is greater than or equal to 2, then M^6 is equivariantly diffeomorphic to a torus manifold.*
(2) *If the rank of the largest singular isotropy subgroup is equal to 1, then M^6 is equivariantly diffeomorphic to $S^3 \times S^3$.*

It remains to prove Part (2) of Theorem 3.2. We have two cases to consider: Case (1), where the circle acts fixed point homogeneously and the induced T^2-action on the codimension two fixed point set is free or almost free and Case (2), where there are isolated T^2 orbits. The remainder of this section will be devoted to the proof of these two cases.

3.1. Proof of Case 1 of Part 2 of Theorem 3.2.

We consider Case (1), where some circle acts fixed point homogeneously and the induced T^2 action on the codimension two fixed point set is either free or almost free. We will prove the following theorem.

Theorem 3.3. *Let T^3 act isometrically and effectively on M^6, a closed, simply-connected, non-negatively curved Riemannian manifold. Suppose that the action is S^1-fixed point homogeneous and that the largest isotropy subgroup of the T^3-action is of rank one. Then M is equivariantly diffeomorphic to $S^3 \times S^3$.*

The strategy for the proof of Theorem 3.3 will be to show that M^6 decomposes as a union of two disk bundles, each a 2-disk bundle over a 4-manifold. One can then show, using classification work of [25], that M^6 has the homology groups of $S^3 \times S^3$ and by Corollary 2.6, we then obtain a diffeomorphism classification. In order to show the equivariant diffeomorphism, we will need to prove that the 4-manifolds are equivariantly diffeomorphic to $S^1 \times S^3$ and use Lemma 2.9.

We begin by establishing some notation. Let F be the fixed point set of the circle action on M^6 and let N be as in Theorem 2.8 such that M^6 is given as

$$M = D(F) \cup_E D(N),$$

where E is the common boundary of the two disk bundles. Observe that F is a closed, orientable, non-negatively curved 4-dimensional submanifold of M^6, admitting an isometric T^2 action. Among other things, we will show in Proposition 3.6 that under these hypotheses, N is also 4-dimensional.

Remark 3.4. *For the remainder of this subsection, we will always assume that there is a T^3 isometric and effective action on M^6, a closed, simply-connected, non-negatively curved Riemannian manifold, such that the action is S^1-fixed point homogeneous and the largest isotropy subgroup of the T^3-action is of rank one. As such, we will omit the statement of these hypotheses in what follows.*

Recall that the rank of a finitely generated abelian group corresponds to the number of $\mathbb{Z}$ factors in the group.

Definition 3.5. *We define the* abelian rank *of a finitely presented group, G, to be the rank of the largest finitely generated abelian subgroup of G and denote it by $rk_{ab}(G)$.*

The following proposition shows that the topology of F and N is restricted when M^6 is S^1-fixed point homogeneous.

Proposition 3.6. *Let M' denote either F or N. Then the following are true:*
1. *$rk(H_1(M')) = 1$;*
2. *$rk_{ab}(\pi_1(M')) \geq 1$;*
3. *$\chi(M') = 0$; and*
4. *M' is orientable.*
5. *$dim(M') = 4$.*

Proof. We will first prove the proposition holds for $M' = F$. If we assume that $\chi(F) \neq 0$, then the induced T^2 action on F would have non-empty fixed point set and thus there is a point in M^6 fixed by T^3, contrary to our hypothesis that the isotropy subgroups have rank at most 1. Thus, $\chi(F) = 0$.

It follows from Lemma 2.10 that $\text{rk}(H_1(F)) \leq 1$. Suppose then that $\text{rk}(H_1(F)) = 0$, to obtain a contradiction. F is orientable, since it is a fixed point set of a circle action and M^6 is orientable. Therefore $\chi(F)$ is strictly positive, a contradiction. Thus $\text{rk}(H_1(F)) = 1$.

Since F is totally geodesic, it has non-negative curvature. Applying the Splitting Theorem of Cheeger and Gromoll [2], we obtain two exact sequences involving $\pi_1(F)$ and $\pi_1(F)/\Gamma$, where Γ is a finite, normal subgroup of $\pi_1(F)$. The first tells us that $\pi_1(F)$ surjects onto $\pi_1(F)/\Gamma$ and the second tells us that $\text{rk}_{ab}(\pi_1(F)/\Gamma) \geq 1$ and so $\text{rk}_{ab}(\pi_1(F)) \geq 1$. Thus the proposition holds for $M' = F$.

We will now show that the proposition holds for $M' = N$. We will first show that $\dim(N) = 4$. Since M^6 decomposes as a union of disk bundles over F and N, respectively, and M^6 is simply-connected, from the Mayer Vietoris sequence of the triple (M, F, N), we have the following long exact sequence:

$$\cdots \to H_1(E) \to H_1(F) \oplus H_1(N) \to 0. \tag{3.1}$$

Now assume N is not 4-dimensional. Since N is the base of a sphere (and not a circle) bundle with total space E, applying the Gysin sequence yields $H_1(E) \cong H_1(N)$. This combined with the fact that Part (1) of the proposition holds for F gives a contradiction to the fact that the map in Display (3.1) is onto. Hence N is 4-dimensional.

Note that by Lemma 2.9, N is T^3-invariant. We have two cases to consider: Case (1), N is not fixed by any circle subgroup of T^3 and Case (2), N is fixed by some circle subgroup of T^3. Note that Case (2) is immediate from the proof of the proposition for F.

It remains to consider Case (1), that is where N is not fixed by any circle subgroup. Then, since it is invariant under the T^3 action, it follows that it is a cohomogeneity one submanifold and hence diffeomorphic to $S^1 \times N^3$ (see Pak [26] and Parker [27]). Recall by Lemma 2.10 that $\text{rk}(H_1(N)) \leq 1$, so by the Künneth formula, it follows that $H_1(N^3)$ is finite. Hence N^3 is one of S^3 or $L_{p,q}$ (see Mostert [19] and Neumann [20]). In particular, N is an orientable submanifold with $\text{rk}(H_1(N)) = \text{rk}_{ab}(\pi_1(N)) = 1$ and $\chi(N) = 0$. □

The next proposition will allow us to narrow down the possibilities for the fundamental groups of F and N. We will assume for this proposition that N is also the fixed point set of some circle action, since in the other case, we already know its fundamental group.

Proposition 3.7. *Let M^4 be F or N and further assume that N is the fixed point set of some circle action. Then*

$$\pi_1(M^4) \cong \mathbb{Z} \quad or \quad \mathbb{Z} \oplus \mathbb{Z}_p \quad or \quad \mathbb{Z} \ltimes \Gamma\,,$$

where Γ is one of the binary dihedral, tetrahedral, octahedral or icosahedral groups.

Proof. By Theorems 12.3 and 12.15 of [3], M^4 is the total space of a fibration with fiber M^3 and base S^1. Then by the homology spectral sequence for fibrations with non-simply-connected base space, we get the following exact sequence:

$$H_2(M) \longrightarrow H_2(S^1) \longrightarrow H_0(S^1; H_1(M^3)) \longrightarrow H_1(M^4) \longrightarrow H_1(S^1) \longrightarrow 0$$

By the Universal Coefficient theorem

$$H_0(S^1; H_1(M^3)) \cong H_0(S^1) \otimes H_1(M^3) \cong \mathbb{Z} \otimes H_1(M^3) \cong H_1(M^3).$$

Now $\mathrm{rk}(H_1(M^4)) = 1$ by Proposition 3.6 and hence $H_1(M^4) = \mathbb{Z} \oplus T_1(M^4)$, where $T_i(M^4)$ is the torsion subgroup of $H_i(M^4)$. Hence the exact sequence becomes

$$0 \longrightarrow H_1(M^3) \longrightarrow \mathbb{Z} \oplus T_1(M^4) \longrightarrow \mathbb{Z} \longrightarrow 0$$

But this implies that $H_1(M^3)$ is finite. Thus only cases 2, 3 and 4 of Proposition 2.7 occur. For cases 2 and 3, since the fundamental group is abelian, $\pi_1(M^4)$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_p$.

In Case 4, $\pi_1(M^3)$ is finite and non-abelian. Since $\pi_2(S^1) = 0$, the long exact sequence in homotopy of $M^3 \hookrightarrow M^4 \to S^1$ is short exact at π_1. This short exact sequence splits and hence $\pi_1(M^4) = \mathbb{Z} \ltimes \Gamma$, where Γ is one of the binary dihedral, tetrahedral, octahedral or icosahedral groups.

□

We are now able to compute the homology groups of N and F.

Proposition 3.8. *Let M^4 denote either F or N. Then $H_1(M^4) = \mathbb{Z}$ and hence $H_i(M^4) \cong H_i(S^1 \times S^3)$ for all i.*

Proof. It follows from Proposition 3.7 that $H_1(M^4) = \mathbb{Z} \oplus T_1(M^4)$, where $T_1(M^4)$ is cyclic or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since both N and F are closed, orientable 4-manifolds, using Poincaré duality, the Universal Coefficient theorem and the fact that $\chi(N) = \chi(F) = 0$, it follows that $\beta_2 = 0$. Hence we obtain:

(1) $H_1(M^4) = \mathbb{Z} \oplus T_1(M^4)$,
(2) $H_2(M^4) = T_1(M^4)$,
(3) $H_i(M^4) = \mathbb{Z}$ for $i \in \{0, 3, 4\}$,

Without loss of generality, we will assume that $T_1(F^4)$ is non-trivial. It follows from the Gysin sequence associated to the circle bundle with total space E and base F^4 that $H^4(E) \cong \mathbb{Z}^2 \oplus T_3(E)$. By duality, we have that $H^1(E) = \mathbb{Z}^2$, $H^2(E) = \mathbb{Z}^{\beta_2(E)} \oplus T_1(E)$ and $H^3(E) = \mathbb{Z}^{\beta_3(E)} \oplus T_2(E)$, noting that $\beta_2 = \beta_3$. Moreover, we can conclude that $|T_1(E)| \leq |T_1(M^4)|$.

Using this information about the cohomology groups of E, we consider the two possibilities for $T_1(E)$: either it is trivial, or it is not. For the first possibility, we immediately obtain a contradiction to the exactness of the Mayer-Vietoris sequence in Display 3.1, as $T_1(F^4)$ is assumed to be non-trivial. For the second possibility, we first consider the case where $T_1(N^4)$ is trivial. However, it follows directly from the Gysin sequence of the associated circle bundle, $S^1 \hookrightarrow E \to N^4$, that $T_1(E)$ is then trivial and thus this case does not occur. So, we are left with one last case: both $T_1(N)$ and $T_1(F)$ are non-trivial. Recall that $|T_1(E)| \leq |T_1(M^4)|$ and we once again obtain a contradiction to the exactness of the Mayer-Vietoris sequence in Display 3.1. Hence $T_1(M^4) = 0$ and $H_i(M^4) \cong H_i(S^1 \times S^3)$ for all i, as desired.

□

We can now determine the fundamental group of M^4.

Lemma 3.9. *Let M^4 denote either F or N. Then $\pi_1(M^4) \cong \mathbb{Z}$.*

Proof. Since $H_1(M^4) = \mathbb{Z}$, it then follows by combining Propositions 3.6 and 3.7, and Proposition 3.3 in Gonçalves and Guaschi [11] that $\pi_1(M^4)$ can only be $\mathbb{Z}$ or $\mathbb{Z} \ltimes I^*$.

We first assume $\pi_1(M^4) \cong \mathbb{Z} \ltimes I^*$ to derive a contradiction. It follows by Theorem 2.11 that $\pi_1(E) \cong \mathbb{Z}^2$. By the long exact sequence in homotopy of the fibration $S^1 \hookrightarrow E \to M^4$, we obtain the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \ltimes I^* \longrightarrow 0,$$

which is not possible. Hence $\pi_1(M^4) \cong \mathbb{Z}$. □

We can now prove the following proposition, which tells us that M^6 has the same homology groups as $S^3 \times S^3$.

Proposition 3.10. *The homology groups of M^6 are isomorphic to those of $S^3 \times S^3$, that is,*

$$H_i(M^6) \cong H_i(S^3 \times S^3) \text{ for all } i.$$

Proof. Consider the Mayer Vietoris sequence of the disk bundle decomposition for M^6. Using Poincaré Duality and the Universal Coefficient Theorem, one immediately concludes that $H_2(M^6) \cong H_4(M^6) = 0$ and that $H_3(M^6)$ has no torsion. So the only unknown homology group is $H_3(M^6)$. Using the Gysin sequence, we see that $\mathrm{rk}(H_3(E)) \leq 1$. Further, using the Universal Coefficient Theorem, it follows that $H_3(E)$ has no torsion, thus $H_3(E)$ is either trivial or $\mathbb{Z}$, and that $H_3(E) \cong \mathrm{Hom}(H_2(E); \mathbb{Z})$. We then have the following exact sequence from the Mayer Vietoris sequence:

$$0 \to H_3(E) \to \mathbb{Z}^2 \to H_3(M^6) \to H_2(E) \to 0.$$

Now, considering the two possibilities for $H_3(E)$, we find that in both cases $H_3(M^6) = \mathbb{Z}^2$. □

Combining the result of Proposition 3.10 with the fact that $\omega_2 = 0$, it follows by Corollary 2.6 that M^6 is diffeomorphic to $S^3 \times S^3$.

We are now in a position to prove Theorem 3.3.

Proof of Theorem 3.3. By Lemma 3.9 and the classification work of [25], it follows that both N and F are T^2-equivariantly diffeomorphic to $S^1 \times S^3$. Recall that circle bundles over a base B are classified by their Euler class $e \in H^2(B)$. Since E is a circle bundle over $S^1 \times S^3$, it is therefore a trivial bundle and hence $E = T^2 \times S^3$. By the classification work of [25], it follows that E is T^3-equivariantly diffeomorphic to $T^2 \times S^3$.

We now have by Lemma 2.9 that M^6 is T^3-equivariantly diffeomorphic to

$$D^2(S^1 \times S^3) \cup_{T^2 \times S^3} D^2(S^1 \times S^3).$$

Since a T^3 effective, isometric action on $S^3 \times S^3$ that has only rank 1 isotropy and is S^1-fixed point homogeneous decomposes exactly as above, we have thus shown that M^6 is T^3-equivariantly diffeomorphic to $S^3 \times S^3$. □

3.2. **Proof of Case 2 of Part 2 of Theorem 3.2.** We now consider the case where there is only isolated circle isotropy, that is where the rank of the isotropy subgroups is at most one and the action is not S^1-fixed point homogeneous. The goal of this subsection is to prove the following theorem.

Theorem 3.11. *Let T^3 act on M^6, a 6-dimensional, closed, simply-connected, non-negatively curved Riemannian manifold. Suppose that the action admits only isolated circle isotropy. Then M^6 is equivariantly diffeomorphic to $S^3 \times S^3$.*

The argument is a straightforward generalization of the finite isotropy case for isometric T^2 actions on closed, simply-connected, non-negatively curved 5-manifolds with only isolated circle orbits that appears in [10]. We include it here for the sake of completeness.

First recall from Corollary 4.7 of Chapter IV of Bredon [1], that the quotient space, M^*, of a cohomogeneity three G-action on a compact, simply-connected manifold with connected orbits is a simply-connected 3-manifold with or without boundary. Note that when there is only isolated circle isotropy for a cohomogeneity three torus action, the quotient space will not have boundary and thus, by the resolution of the Poincaré conjecture (see Perelman [28, 29, 30]), we have that $M^* = S^3$.

We first recall Proposition 4.5 from [10], which gives us a lower bound for the number of isolated singular orbits of the action.

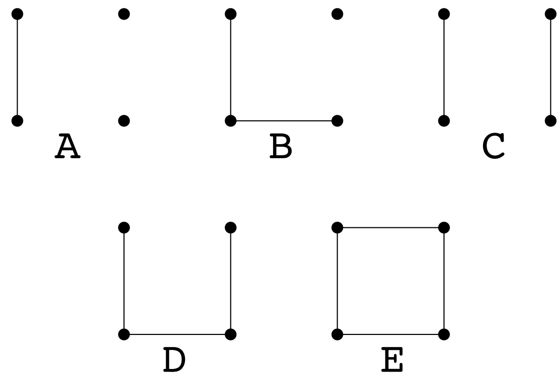


FIGURE 3.1. Possible weighted graphs when there is finite cyclic isotropy.

Proposition 3.12. [10] *Let T^n act on M^{n+3}, a simply-connected, smooth manifold. Suppose that M^* is homeomorphic to S^3 and that there are exactly two orbit types: principal orbits T^n and isolated singular orbits T^{n-1}. Then there are at least $n+1$ isolated singular orbits T^{n-1}.*

The non-negative curvature hypothesis gives us an upper bound on the number of isolated T^2 orbits. The following lemma from [13] is crucial:

Lemma 3.13. [13] *A three dimensional non-negatively curved Alexandrov space X^3 has at most four points for which the space of directions is not larger than $S^2(1/2)$.*

Proposition 4.8 in [10] shows that if there is finite isotropy, it must be $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_k$ and that in the latter case, those exceptional orbits are not isolated. Combining Proposition 3.12 and Lemma 3.13 it follows that there are exactly 4 isolated T^2 orbits. This result combined with the proof of Proposition 5.8 in [10] then tells us that $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ isotropy cannot occur.

We may summarize our results as follows.

Proposition 3.14. *Let T^3 act isometrically and effectively on M^6, a 6-dimensional, closed, simply-connected Riemannian manifold as in Theorem 3.11. Suppose that $M^6/T^3 = M^* = S^3$. Then there are exactly 4 isolated T^2 orbits and if there is finite isotropy, then it must be cyclic and the corresponding orbits are not isolated.*

We consider first the case where there is no finite isotropy. We have the following result.

Proposition 3.15. *Let T^3 act isometrically and effectively on M^6, a 6-dimensional, closed, simply-connected Riemannian manifold. Suppose that $M^6/T^3 = M^* = S^3$. If there is no finite isotropy, then M^6 is diffeomorphic to $S^3 \times S^3$.*

Proof. The proof of Proposition 3.12 shows that $\pi_2(M^6) = 0$. By the Hurewicz isomorphism, it follows that M^6 only has homology in dimension 3 and by the Universal Coefficients there is no torsion. Since the fixed point set of the T^3-action is empty by hypothesis, it follows that $\chi(M^6)$=0. This tells us that $b_3(M^6) = 2$ and thus M^6 has the homology groups of $S^3 \times S^3$, so by Corollary 2.6, it follows that M^6 is diffeomorphic to $S^3 \times S^3$. □

We now consider the case where the T^3-action on M^6 has non-trivial finite isotropy. There are just five admissible graphs corresponding to this case (see Figure 3.1).

In the special case where the singular set in the orbit space contains a circle we have the following result which follows directly from work of [13] and its generalization in [10].

Theorem 3.16. *Let M^6 be a closed, simply-connected, non-negatively curved 6-manifold with an isometric T^3 action and orbit space $M^* \simeq S^3$. If the singular set in the orbit space M^* contains a circle K^1, then the following hold:*

(1) *The circle K^1 is the only circle in the singular set in M^*.*

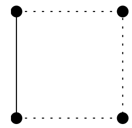


FIGURE 3.2. How to complete a weighted graph with edges corresponding to principal orbits to obtain a circle: the solid edge corresponds to orbits with finite cyclic isotropy, while the dotted edges correspond to principal orbits.

(2) *K^1 comprises all of the singular set, i.e., $M^* \setminus K^1$ is smooth.*
(3) *The circle K^1 is unknotted in M^*.*

We will now show in all cases where we have a circle that we may decompose the manifold as a union of disk bundles, where at least one of the disk bundles is over one arc of the circle.

Proposition 3.17. *Let T^3 act on M^6 isometrically and effectively and suppose that $M^* = S^3$ and there is finite isotropy. Suppose further that the singular set in S^3 corresponds to graph E in figure 3.1. Then we may decompose M^6 as a union of disk bundles over two disjoint 4-dimensional submanifolds fixed by finite isotropy (although not necessarily the same group).*

The proof of this proposition is exactly the same as in [10] (see the proofs of Proposition 6.9 and Proposition 6.7(2)). For graphs (A) through (D), we may complete the weighted graph by joining disjoint isolated circle orbits or arcs via edges corresponding to shortest geodesics consisting of regular points in the orbit space. In this way we obtain a graph that is an unknotted circle (see figure 3.2) and now for all the possible graphs we may decompose M^6 as the union of two disk bundles over the 4-dimensional manifolds that correspond to opposite arcs of the circle. These 4-dimensional manifolds are invariant under the T^3 action and via the classification of torus actions of cohomogeneity one (see [27, 26]), it follows that they are $T^1 \times M^3$, where M^3 is an orientable, cohomogeneity one manifold equal to one of S^3, $L_{p,q}$, $S^2 \times S^1$ by [19] and [20]. By Lemma 2.10, it follows that the 4-dimensional manifold may be one of $S^1 \times S^3$ or $S^1 \times L_{p,q}$.

As in Case (1) of Part 2 of Theorem 3.2, analyzing the Mayer-Vietoris sequence of the decomposition it is immediate that the 4-dimensional manifolds corresponding to opposite arcs for all the graphs must be $S^1 \times S^3$ and M^6 has the homology groups of $S^3 \times S^3$. Applying Corollary 2.6, it follows that M^6 is diffeomorphic to $S^3 \times S^3$.

It remains to show that the classification is up to equivariant diffeomorphism in both cases. The argument in the proof of Proposition 3.16 (see [13] and [10]) uses the construction of a vector field V^* on M^*. We construct V^* so that the flow lines emanating from each point of one edge will meet at a point of the other edge to form a 2-sphere, unless the points are vertices of the rectangle, in which case there is only one flow line. Moreover, there is an S^1 action on $M^* = S^3$ preserving these spheres with orbit space a 2-dimensional rectangle. This action clearly lifts to an action on M whose orbits near the two 4-dimensional submanifolds are the normal circles in a tubular neighborhood. It follows that this lift commutes with the given isometric T^3-action on M^6. Thus the T^3-action on M^6 extends to a smooth T^4-action.

The following theorem will allow us to apply Theorem 2.5 in Oh [21] which states the following. If the matrix of the circle isotropy subgroups of a T^4 action on M^6 has determinant ± 1, then M^6 is equivariantly diffeomorphic to $S^3 \times S^3$. Hence Theorem 3.18 will complete the proof of Case (2) Theorem 3.2.

Theorem 3.18. *Let T^{n-k} act effectively on M^n such that $M^n/T^{n-k} = D^k$, $k \geq 2$. Further assume that all singular isotropy is connected, all singular orbits correspond to boundary points and that there are no exceptional orbits. Then M^n is simply-connected if and only if there are $(n-k)$ distinct circle isotropy groups whose matrix has determinant ± 1.*

Proof. First assume that M^n is simply connected. Corollary 2.9 in [6], which generalizes a result in [16], says that with the above hypotheses the isotropy subgroups of the T^{n-k} action generate T^{n-k}, and there are at least $n-k$ distinct circle isotropy subgroups. Let Δ be the matrix of the $(n-k)$ distinct circle isotropy groups. It is shown that the $n-k$ isotropy subgroups of the T^{n-k} action generate T^{n-k} if and only $\det(\Delta) = \pm 1$ in Lemma 1.4 in [23].

The converse is proven for $k = 2$ in Corollary 1.2 in Oh [22]. We can generalize the result to $k \geq 2$ by observing that the proof only requires that the regular part of the manifold be $\mathring{D}^k \times T^{n-k}$. Hence we see that if $\det(\Delta) = \pm 1$, then M^n is simply-connected.

□

Remark 3.19. *Note that Theorem 3.18 is optimal, since for $k = 1$ there are cohomogeneity one T^1 actions on $\mathbb{R}P^2$ as well as cohomogeneity one T^2 actions on $L(p, 1)$ such that* $\det(\Delta) = \pm 1$.

References

[1] G. Bredon, *Introduction to compact transformation groups*, Academic Press 48 (1972).

[2] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971/72), 119–128.

[3] P. Conner, F. Raymond, *Holomorphic Seifert fiberings*, Proc. Second Conference in Transformation Groups, Lecture Notes in Mathematics, no. 299, Springer, Berlin (1972), 124–204.

[4] J. Davis and P. Kirk, *Lecture Notes in Algebraic Topology*, Graduate Studies In Mathematics, Book 35, Am. Math. Soc. (2001).

[5] J. DeVito, *The classification of compact simply connected biquotients in dimension* 6 *and* 7, Math. Ann., DOI 10.1007/s00208-016-1460-8 (2016).

[6] C. Escher and C. Searle, *Non-negative curvature and torus actions*, arXiv preprint, arXiv:1506.08685v2[math.DG] 13 Apr 2016.

[7] F. Fang and X. Rong, *Homeomorphism Classification of Positively Curved Manifolds with Almost Maximal Symmetry Rank*, Math. Ann. **332** no. 1 (2005) 81–101.

[8] F. Galaz-García and M. Kerin, *Cohomogeneity two torus actions on non-negatively curved manifolds of low dimension*, Math. Zeitschrift **276** (1-2) (2014), 133–152.

[9] F. Galaz-Garcia and C. Searle, *Low-dimensional manifolds with non-negative curvature and maximal symmetry rank*, Proc. Amer. Math. Soc. **139** (2011) 2559–2564.

[10] F. Galaz-Garcia and C. Searle, *Nonnegatively curved* 5*-manifolds with almost maximal symmetry rank*, Geom. Topol. **18** (2014), no. 3, 1397–1435.

[11] D. L. Gonçalves, J. Guaschi, *The lower central and derived series of the braid groups of the sphere*, Transactions of the AMS, **361** (2009), 3375–3399.

[12] K. Grove and C. Searle, *Positively curved manifolds with maximal symmetry rank*, Jour. of Pure and Appl. Alg. **91** (1994) 137–142.

[13] K. Grove and B. Wilking, *A knot characterization and* 1*-connected nonnegatively curved 4- manifolds with circle symmetry*, Geom. Topol. **18** no. 5 (2014), 3091–3110.

[14] H. Ishida, *Complex manifolds with maximal torus actions*, arXiv:1302.0633v3 30 Apr 2015.

[15] P. E. Jupp, *Classification of certain* 6*-manifolds*, Mathematical Proceedings of the Cambridge Philosophical Society **73** no. 2 (1973) 293–300.

[16] S. Kim, D. McGavran and J. Pak, *Torus group actions in simply connected manifolds*, Pacific J. Math., **53** no.2 (1976), 435–444.

[17] B. Kleiner, PhD thesis, U.C. Berkeley (1989).

[18] S. Kuroki, *An Orlik-Raymond type classification of simply-connected six-dimensional torus manfiolds with vanishing odd degree cohomology*, Pac. J. Math **280** no. 1 (2016), 89–114.

[19] P. S. Mostert, *On a compact Lie group acting on a manifold*, Ann. of Math., **65** no. 2 (1957), 447-455.

[20] W. D. Neumann, *3-dimensional G-manifolds with 2-dimensional orbits*, 1968 Proc. Conf. on Transformation Groups (New Orleans, La., 1967), Springer, New York, 220–222.

[21] H. S. Oh, 6*-dimensional manifolds with effective T^4 actions*, Topology Appl. 13 (1982), no. 2, 137–154.

[22] H. S. Oh, Toral actions on 5- and 6-dimensional manifolds, PhD thesis, U. MIchigan (1980).

[23] H. S. Oh, *Toral actions on* 5*-manifolds*, Transactions of the AMS **278** no. 1 (1983) 233–252.

[24] P. Orlik, F. Raymond, *Actions of $SO(2)$ on* 3*-manifolds*, Proc. Conf. Transformation Groups, Springer, New York (1968), 297–318.

[25] P. Orlik, F. Raymond, *Actions of the torus on* 4*-manifolds, II*, Topology, **13** (1974), 89–112.

[26] J. Pak, *Actions of torus T^n on $(n+1)$-manifolds M^{n+1}*. Pacific J. Math., **44**, no. 2 (1973), 671–674.

[27] J. Parker, *4-dimensional G-manifolds with 3-dimensional orbit*, Pacific J. Math, **125**, no. 1 (1986), 187–204.

[28] G. Perelman, "The entropy formula for the Ricci Flow and its geometric applications", arXiv:math.DG/0211159, 11 Nov 2002.

[29] G. Perelman, "Ricci Flow with surgery on three-manifolds", arXiv:math.DG/0303109, 10 Mar 2003.

[30] G. Perelman, "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds", arXiv:math.DG/0307245, 17 Jul 2003.

[31] X. Rong, *Positively curved manifolds with almost maximal symmetry rank*, Geom. Ded. **95** (2002) 157–182.

[32] C. Searle and D. Yang, *On the topology of non-negatively curved simply-connected 4-manifolds with continuous symmetry*, Duke Math. J. **74** no. 2 (1994) 547–556.

[33] W. Spindeler, *S^1-actions on* 4*-manifolds and fixed point homogeneous manifolds of nonnegative curvature*, PhD Thesis, Westfälische Wilhelms-Universität Münster (2014).

[34] C. T. C. Wall, *Classification problems in differential topology - IV*, Topology **6** (1967) 273–296.

[35] M. Wiemeler, *Torus manifolds and non-negative curvature*, J. London Math. Soc. (2015) **91** no. 3, 667-692.

[36] B. Wilking, *Torus actions on manifolds of positive sectional curvature*, Acta Math., **191** no. 2 (2003) 259–297.

[37] A. V. Zhubr, *A decomposition theorem for simply connected* 6*-manifolds*, LOMI seminar notes **36** (1973) 40–49 (Russian).

[38] A. V. Zhubr, *Classification of simply connected six-dimensional spinor manifolds*, (English) Math. USSR, Izv. **9** (1975), (1976), 793–812.

[39] A. V. Zhubr, *Closed simply connected six-dimensional manifolds: proofs of classification theorems*, Algebra i Analiz **12** (2000), no.4, 126–230.

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