TORUS ACTIONS, MAXIMALITY AND NON-NEGATIVE CURVATURE

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Abstract. Let $\mathcal{M}_0^n$ be the class of closed, simply-connected, non-negatively curved Riemannian manifolds admitting an isometric, effective, almost maximal or maximal torus action. We prove that if $M \in \mathcal{M}_0^n$, then M is equivariantly diffeomorphic to the free linear quotient by a torus of a product of spheres of dimensions greater than or equal to three. As an immediate consequence, we prove the Maximal Symmetry Rank Conjecture for all $M \in \mathcal{M}_0^n$. Finally, we show the Maximal Symmetry Rank Conjecture for simply-connected, non-negatively curved manifolds holds for dimensions less than or equal to nine without assuming the torus action is almost maximal or maximal.

1. Introduction

The classification of compact Riemannian manifolds with positive or non-negative sectional curvature is a long-standing problem in Riemannian geometry. One successful approach has been the introduction of symmetries, and an important first case to understand is that of continuous abelian symmetries, that is, of torus actions. Our main result, stated below in Theorem A, gives an equivariant diffeomorphism classification of closed, simply connected, non-negatively curved Riemannian manifolds with an isometric, effective and maximal or almost maximal torus action. This result significantly strengthens a recent classification up to equivariant rational homotopy equivalence of a subclass of these manifolds (cf. Theorems D and B in Galaz-García, Kerin, Radeschi, and Wiemeler [14]).

One defines a T^k action on a smooth manifold, M^n, to be *maximal* (respectively, *almost maximal*), when $2k - n$ (respectively, $2k - n + 1$) is equal to the dimension of the smallest orbit. Torus manifolds are examples of maximal T^k-actions with 0-dimensional smallest orbit. We obtain the following equivariant diffeomorphism classification.

Theorem A. *Let T^k act isometrically and effectively on M^n, a closed, simply-connected, non-negatively curved Riemannian manifold. Assume that the action is almost maximal or maximal. Then M is equivariantly diffeomorphic to the free linear quotient of $\mathcal{Z}$,*

$$\mathcal{Z} = \prod_{i<r} S^{2n_i} \times \prod_{i \geq r} S^{2n_i - 1},\ n_i \geq 2,$$

a product of spheres of dimensions greater than or equal to 3 and with $n \leq dim(\mathcal{Z}) \leq 3n - 3k$.

Note that for closed, simply-connected manifolds of strictly positive curvature, a maximal action can only occur when the rank is approximately half the dimension of the manifold and this corresponds to the maximal symmetry rank case, where the symmetry rank of a manifold is defined as the rank of the isometry group of M. In particular, by work of Grove and Searle [17], such manifolds are, up to equivariant diffeomorphism, spheres or complex projective spaces.

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In contrast, the corresponding maximal symmetry rank for manifolds of non-negative curvature was conjectured by Galaz-García and Searle [15, 16] to be approximately two-thirds the dimension of the manifold. In [15], they obtained a diffeomorphism classification for the non-negative curvature case, but only in dimensions less than or equal to 6. Based on our results here, we reformulate and sharpen the conjecture for non-negative curvature (cf. [15]).

Maximal Symmetry Rank Conjecture. *Let T^k act isometrically and effectively on M^n, a closed, simply-connected, non-negatively curved Riemannian manifold. Then*

(1) $k \leq \lfloor 2n/3 \rfloor$;
(2) *When $k = \lfloor 2n/3 \rfloor$, M^n is equivariantly diffeomorphic to*

$$Z = \prod_{i \leq r} S^{2n_i+1} \times \prod_{i>r} S^{2n_i}, \quad \text{with } r = 2\lfloor 2n/3 \rfloor - n,$$

or the quotient of Z by a free linear action of a torus of rank less than or equal to $2n$ mod 3.

We will show that the upper bound on the rank of the action in Part (1) of the Maximal Symmetry Rank Conjecture is exactly the upper bound on the rank of a maximal or almost maximal action on a closed, simply-connected, non-negatively curved Riemannian manifold. Combining this result with the lower bound for a maximal action we obtain the following corollary of Theorem A.

Corollary B. *Let T^k act isometrically and effectively on M^n, a closed, simply-connected, non-negatively curved Riemannian manifold. Assume that the action is almost maximal or maximal. Then $\lfloor (n+1)/2 \rfloor \leq k \leq \lfloor 2n/3 \rfloor$.*

Thus, there are two extremal cases for Theorem A: the case of torus manifolds, where the rank of the action is half the dimension of the manifold, and the case of those with maximal symmetry rank, which corresponds to the case when the rank is approximately two-thirds the dimension of the manifold.

Observation. *There are examples of closed, simply-connected, complex manifolds, known as LVMB manifolds, of complex dimension $n - m$, with $2m \leq n$, admitting rank n maximal torus actions, with $n > \lfloor 2(2n-2m)/3 \rfloor$ (see Ishida [23]). Corollary B tells us that these manifolds cannot admit non-negative sectional curvature.*

As a direct application of Theorem A in combination with Lemma 2.2 from [23] (Lemma 2.3 in this article), we obtain the Maximal Symmetry Rank conjecture in the presence of a maximal or almost maximal action.

Theorem C. *Let T^k act isometrically and effectively on M^n, a closed, simply-connected, non-negatively curved Riemannian manifold. Assume that the action is almost maximal or maximal for $k = \lfloor 2n/3 \rfloor$. Then the Maximal Symmetry Rank Conjecture holds.*

In fact, the proof of Theorem C tells us that we may reformulate the Maximal Symmetry Rank Conjecture as follows.

Maximal Symmetry Rank Conjecture. *Let T^k act isometrically and effectively on M^n, a closed, simply-connected, non-negatively curved Riemannian manifold. Then the action is maximal for $k = \lfloor 2n/3 \rfloor$.*

Our results also have implications for the class of rationally elliptic manifolds. In Proposition 5.4, we find a lower bound for the free rank of a T^k action, where the *free rank* is the dimension of the largest subtorus that can act almost freely. Combining this result with the upper bound for the free rank obtained in Galaz-García, Kerin and Radeschi [13], allows us to reformulate Theorem C as follows:

Theorem C′. *Let T^k act isometrically and effectively on M^n, a closed, simply-connected, non-negatively curved Riemannian manifold that is rationally elliptic. Then the Maximal Symmetry Rank Conjecture holds.*

Observe that if the Bott Conjecture were true, then we would no longer need to assume rational ellipticity in Theorem C′ nor that the action is maximal or almost maximal in Corollary B. That is, the Maximal Symmetry Rank Conjecture would hold for all manifolds, M^n, that are simply-connected, closed, non-negatively curved and admit an isometric T^k action.

As mentioned above, Theorem A gives us important information for closed, simply-connected, non-negatively curved manifolds of maximal symmetry rank. Recall first that such manifolds have been classified up to diffeomorphism in dimensions less than or equal to 6 in [15] and up to equivariant diffeomorphism in dimensions 4 through 6 by Galaz-García and Kerin [12]. It is then of interest to extend this classification to higher dimensions. In Proposition 7.1, we show that a cohomogeneity three torus action must be maximal for this class of manifolds, provided the dimension is greater than or equal to 7. We can then apply Theorem A to obtain the following classification theorem.

Theorem D. *Let M^n, $n = 7, 8, 9$ be a closed, simply-connected, non-negatively curved manifold admitting an effective, isometric torus action. Then the Maximal Symmetry Rank Conjecture holds.*

Finally, as an immediate consequence of Corollary B combined with Proposition 7.1, we confirm Part (1) of the Maximal Symmetry Rank Conjecture for dimensions less than or equal to 12.

Corollary E. *Let M^n, $n \leq 12$, be a closed, simply-connected, non-negatively curved Riemannian manifold admitting an effective, isometric torus action. Then Part (1) of the Maximal Symmetry Rank Conjecture holds.*

It is also of interest to classify closed, simply-connected, non-negatively curved Riemannian n-dimensional manifolds of almost maximal symmetry rank, that is, admitting a T^k isometric, effective action of rank $k = \lfloor 2n/3 \rfloor - 1$. In a separate article [10], the authors will use Theorem A in combination with results about T^3 actions with only circle isotropy to obtain a classification of 6-dimensional, closed, simply-connected, non-negatively curved manifolds of almost maximal symmetry rank, thereby extending the almost maximal symmetry rank classification work of Kleiner [25] and Searle and Yang [45] in dimension 4 and work of Galaz-García and Searle [16] in dimension 5. We expect that Theorem A should admit a number of similar applications and should be particularly useful for any classification of closed, simply-connected, non-negatively curved manifolds of higher dimension of maximal or almost maximal symmetry rank.

1.1. **Organization.** We have organized the paper in general so as to present the topological tools and results first, followed by their geometrical counterparts. In Section 2, we describe the topological and geometrical tools we will need to prove Theorem A, Corollary B and Theorem D. In Section 3, we prove a generalization of the Equivariant Cross-Sectioning Theorem of Orlik and Raymond and thereby obtain an Equivariant Classification Theorem. In Section 4, we generalize results on torus manifolds of non-negative curvature to the class of almost non-negatively curved torus manifolds with non-negatively curved quotient spaces. In Section 5, we find a general lower bound for the free rank of an action on an Alexandrov space with a lower curvature bound and also show that an almost maximal action is actually maximal in the presence of non-negative curvature. In Section 6, we prove Theorem A. In Section 7, we prove Corollary B, Theorem D, and we present a significantly streamlined proof of the Maximal Symmetry Rank Conjecture for dimensions less than or equal to 6.

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2. Preliminaries

In this section we will gather basic results and facts about transformation groups, torus actions, torus manifolds, torus orbifolds, as well as results concerning G-invariant manifolds of non-negative and almost non-negative sectional curvature.

2.1. Transformation Groups.

Let G be a compact Lie group acting on a smooth manifold M. We denote by $G_x = \{ g \in G : gx = x \}$ the *isotropy group* at $x \in M$ and by $G(x) = \{ gx : g \in G \} \simeq G/G_x$ the *orbit* of x. Orbits will be *principal*, *exceptional* or *singular*, depending on the relative size of their isotropy subgroups; namely, principal orbits correspond to those orbits with the smallest possible isotropy subgroup, an orbit is called exceptional when its isotropy subgroup is a finite extension of the principal isotropy subgroup and singular when its isotropy subgroup is of strictly larger dimension than that of the principal isotropy subgroup.

The *ineffective kernel* of the action is the subgroup $K = \cap_{x\in M} G_x$. We say that G acts *effectively* on M if K is trivial. The action is called *almost effective* if K is finite. The action is *free* if every isotropy group is trivial and *almost free* if every isotropy group is finite. As mentioned in the Introduction, the *free rank* of an action is the rank of the maximal subtorus that acts almost freely. In order to further distinguish between the case when the free rank corresponds to a free action and the case when it corresponds to an almost free action, we make the following definition.

Definition 2.1 (**Free Dimension**). *Suppose that the free rank of a T^k-action is equal to r and let T^r denote the maximal subtorus of T^k acting almost freely. We say that the* free dimension *is equal to the dimension of the largest subtorus of T^r that acts freely.*

We will sometimes denote the *fixed point set* $M^G = \{ x \in M : gx = x, g \in G \}$ of the G-action by $\mathrm{Fix}(M; G)$. We define its dimension as

$$\dim(\mathrm{Fix}(M; G)) = \max\{ \dim(N) : N \text{ is a connected component of } Fix(M; G) \}.$$

One measurement for the size of a transformation group $G \times M \to M$ is the dimension of its orbit space M/G, also called the *cohomogeneity* of the action. This dimension is clearly constrained by the dimension of the fixed point set M^G of G in M. In fact, $\dim(M/G) \geq \dim(M^G) + 1$ for any non-trivial action. In light of this, the *fixed-point cohomogeneity* of an action, denoted by $\mathrm{cohomfix}(M; G)$, is defined by

$$\mathrm{cohomfix}(M; G) = \dim(M/G) - \dim(M^G) - 1 \geq 0.$$

A manifold with fixed-point cohomogeneity 0 is also called a *G-fixed point homogeneous manifold.*

Let $G = H \times \cdots \times H = H^l$ act isometrically and effectively on M^n. Let N_1 denote the connected component of largest dimension in M^H and let $N_k \subset N_{k-1}$ denote the connected component of largest dimension in N_{k-1}^H, for $k \leq l$. We will call a manifold *nested H-fixed point homogeneous* when there exists a tower of nested H-fixed point sets,

$$N_l \subset N_{l-1} \subset \cdots \subset N_1 \subset M$$

such that H acts fixed point homogeneously on M^n, and all induced actions of G/H^k on N_k, for all $1 \leq k \leq l-1$, are H-fixed point homogeneous.

2.2. **Torus Actions.** In this subsection we will recall notation and facts about smooth G actions on smooth n-manifolds, M, in the special case when G is a torus. We first recall the definition of a *maximal torus action*, introduced in [23].

Definition 2.2 (**Maximal Action).** *Let M^n be a connected manifold with an effective $G = T^k$ action.*

(1) *We call the G-action on M^n* maximal *if there is a point $x \in M$ such that the dimension of its isotropy group is $n-k$: $dim(G_x) = n-k$.*
(2) *The orbit $G(x)$ through $x \in M$ is called* minimal *if $dim(G(x)) = 2k - n$.*

Note that the action of T^k on M is maximal if and only if there exists a minimal orbit $T^k(x)$. The following lemma of [23] shows that a maximal action on M means that there is no larger torus which acts on M effectively.

Lemma 2.3. [23] *Let M be a connected manifold with an effective T^k action. Let $T^l \subset T^k$ be a subtorus of T^k. Suppose that the action of T^k restricted to T^l on M is maximal. Then $T^l = T^k$.*

A similar statement about almost maximal actions is also true.

Lemma 2.4. *Let M be a connected manifold with an effective T^k action. Let $T^l \subset T^k$ be a subtorus of T^k. Suppose that the action of T^k restricted to T^l on M is almost maximal. Then $l = k$ or $l = k-1$.*

We also obtain the following properties of a maximal action.

Lemma 2.5. [23] *Let M be a connected manifold with a maximal $G = T^k$ action. Let $G(x)$ be a minimal orbit. Then*

(1) *The isotropy group G_x at x is connected.*
(2) *$G(x)$ is a connected component of the fixed point set of the action of T^k restricted to G_x on M.*
(3) *Each minimal orbit is isolated. In particular, there are finitely many minimal orbits if M is compact.*

Observation 2.6. *It is easy to see that Properties (1) and (2) also hold for an almost maximal action.*

2.3. **Orbit Spaces.** Now, to any orbit space, M/G, we may assign isotropy information, in the form of weights. We recall the definition of a weighted orbit space for a smooth G action on M.

Definition 2.7 (**Weighted Orbit Space).** *Let G act smoothly on an n-manifold M with orbit space $M^* = M/G$. To each orbit in M^* there is associated to it a certain orbit type which is characterized by the isotropy group of the points of the orbit together with the slice representation at the given orbit. This orbit space together with its orbit types and slice representation is called a* weighted orbit space*.*

Letting $G = T^k$ act maximally on M^n, we note that it is enough to specify the weights of the *facets*, that is, of the codimension one faces, as these will correspond to circle isotropy groups and together with a description of the orbit space, one then obtains a complete description of all orbit types.

Let

$$\begin{aligned} p \quad : \quad & \mathbb{R}^k \longrightarrow T^k \\ & (x_1, \dots, x_k) \mapsto (e^{2\pi x_1 i}, \dots, e^{2\pi x_k i}) \end{aligned}$$

be the universal covering projection and let G be a circle subgroup of T^k. Since each component of $p^{-1}(G)$ is a line containing at least two integer lattice points of $\mathbb{R}^k$, it is natural to parametrize G as follows: Let $a_1, \dots, a_k$ be relatively prime integers and let $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}^k$. We call the $a_i \in \mathbb{Z}$ the weights of the corresponding circle isotropy subgroup $G(\mathbf{a})$. Then $G(\mathbf{a}) = G(a_1, \dots, a_k) = \{(x_1, \dots, x_k) | x_i = a_i t \mod \mathbb{Z}, 0 \leq t < 1, i = 1, \dots k\}$. With this notation $G(\mathbf{a})$ is the image of a line in $\mathbb{R}^k$ through the origin and the lattice point $\mathbf{a} = (a_1, \dots, a_k)$ under the projection p. We define the matrix of m isotropy groups $G(\mathbf{a}_1), \dots, G(\mathbf{a}_m)$ to be the following $m \times k$-matrix:

$$M_{(\mathbf{a}_1, \dots, \mathbf{a}_m)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \dots & a_{km} \end{bmatrix}$$

We will denote the weights of a torus action via the matrix $M_{(\mathbf{a}_1, \dots, \mathbf{a}_m)}$.

The following proposition and corollary are straightforward generalizations of Theorem 1.6 and Corollary 1.7 in Kim, McGavran and Pak, [24], respectively. We leave the proofs to the reader.

Proposition 2.8. *Let T^k act on M^n effectively, where M^n is a simply-connected, closed manifold of dimension n. Suppose further that all isotropy subgroups are connected, all singular orbits correspond to points on the boundary of the quotient space, there are no exceptional orbits and $M^n/T^k = D^{n-k}$. Then no subgroup T^l $(k > l \geq 1)$ can contain all nonfree elements of T^k.*

Corollary 2.9. *With the same hypotheses as in Proposition 2.8, all isotropy subgroups must generate the whole group T^k and there are at least k different circle isotropy subgroups of T^k.*

We recall the following facts about circle subgroups of a T^k-action on a manifold, M^n, see, for example, Oh [33] for more details.

Proposition 2.10. [33] *Let T^k act on M^n. The following are true.*

1. *Two circle subgroups $G(\mathbf{a}_1)$ and $G(\mathbf{a}_2)$ of T^k have trivial intersection if and only if there exist $\mathbf{a}_3, \dots, \mathbf{a}_k \in \mathbb{Z}^k$ such that the determinant of the corresponding $k \times k$ matrix, $M_{(\mathbf{a}_1, \dots, \mathbf{a}_k)}$, is equal to plus or minus one, that is,* $\det(M_{(\mathbf{a}_1, \dots, \mathbf{a}_k)}) = \pm 1$.
2. *The k circle subgroups generate T^k, that is $G(\mathbf{a}_1) \times \cdots \times G(\mathbf{a}_k) = T^k$ if and only if* $\det(M_{(\mathbf{a}_1, \dots \mathbf{a}_k)}) = \pm 1$.

With the hypotheses as in Proposition 2.8, we obtain the following information about the quotient space $M^n/T^k = D^{n-k}$: Corollary 2.9 tells us that M/T has least k facets and Part (2) of Proposition 2.10 tells us that the matrix of weights for M/T must have determinant ± 1.

Note that the slice representation at a fixed point is just the action of the isotropy group on the normal disk. For a circle orbit of type $G(a_1, \ldots, a_k)$ we have the standard representation described in terms of relatively prime integers. Let M_1^* and M_2^* denote the orbit spaces of a smooth action of G on the closed oriented smooth n-manifolds M_1 and M_2. A homeomorphism (diffeomorphism) of M_1^* and M_2^* which carries the weights of M_1^* isomorphically onto the weights of M_2^* is called a *weight-preserving homeomorphism (diffeomorphism)*.

For classification purposes it is convenient to fix orientations. We start with a fixed orientation of the group G. Then an orientation of M determines an orientation of M^* and vice versa, assuming there are no isotropy groups which reverse the orientation of a slice. When the orbit map $\pi : M \to M^*$ has a *cross-section* s, that is, $s : M^* \to M$ is a continuous map such that $\pi \circ s$ is the identity on M^*, we always assume that the orientation of $s(M^*)$ is the one induced by the cross-section and the orientation of M is compatible with it.

We now recall the definition of a conical orbit structure of a G-action on a space X.

Definition 2.11 (**Conical Orbit Structure**). *Denote by* $\overline{C(Y)} = (Y \times I)/(Y \times \{0\})$ *the closed cone over a space* Y *and by* $C(Y) = Y \times [0, 1)/(Y \times \{0\})$ *the open cone. The orbit structure of* X *is called* conical*, if* X^* *is homeomorphic to an open cone* $C(Y)$ *with constant orbit type along rays, less the vertex,* p^**.*

For the case of a T^k-action on a closed manifold M^n, McGavran in [28] (see also [29]) obtained the following classification of a neighborhood of point in M with isotropy group T^l, $l \leq k$.

Theorem 2.12. [28, 29] *Suppose* T^k *acts locally smoothly on a closed manifold* M^n*. Suppose* $p \in M^n$ *has isotropy group* T^l*,* $0 \leq l \leq k$*. Let* X *be a closed invariant neighborhood of* p *in* M *such that* $X^* = \overline{C(Y)}$*. Suppose* $C(Y)$ *is an open subspace of* M^* *with conical orbit structure with vertex* p^**. Then* X *is equivariantly homeomorphic to* $T^{k-l} \times D^{n-k+l}$*.*

2.4. **Torus Manifolds.** An important subclass of manifolds admitting an effective torus action are the so-called *torus manifolds*. For more details on torus manifolds, we refer the reader to Hattori and Masuda [21], Buchstaber and Panov [4], and Masuda and Panov [27].

Definition 2.13 (**Torus Manifold**). *A* torus manifold M *is a* $2n$*-dimensional closed, connected, orientable, smooth manifold with an effective smooth action of an* n*-dimensional torus* T *such that* $M^T \neq \emptyset$*.*

Note that torus manifolds have the following properties. The action of T^n on M^{2n} is an example of a *maximal* action. Further, M^{2n} is an example of a S^1-fixed point homogeneous manifold and moreover, the action on T^n on M^{2n} is an example of a *nested* S^1*-fixed point homogeneous action*, as defined in Section 2, that is, we can find a tower of nested fixed point sets for each $p \in M^T$:

$$p \subset F^2 \subset \cdots \subset F^{2n-2} \subset M^{2n}.$$

We will also make use of the following refinement of a torus manifold.

Definition 2.14 (**Locally Standard**). *A torus manifold* M^{2n} *is called* locally standard *if each point in* M^{2n} *has an invariant neighborhood* U *which is weakly equivariantly diffeomorphic to an open subset* $W \subset \mathbb{C}^n$ *invariant under the standard* T^n *action on* $\mathbb{C}^n$*, that is, there exists an automorphism* $\psi : T^n \longrightarrow T^n$ *and a diffeomorphism* $f : U \longrightarrow W$ *such that* $f(ty) = \psi(t)\, f(y)$ *for all* $t \in T^n$ *and* $y \in U$*.*

Observe that not all torus manifolds are locally standard. In fact, being locally standard imposes strong topological restrictions, as we see in the following theorem of [27].

Theorem 2.15. [27] *Let M be a torus manifold. Then the odd degree cohomology of M vanishes if and only if M is locally standard and the orbit space M/T is acyclic with acyclic faces.*

The quotient space $P^n = M^{2n}/T^n$ of a torus manifold plays an important role in the theory. Recall that an n-dimensional convex polytope is called *simple* if the number of facets meeting at each vertex is n. An *n-manifold with corners* is a Hausdorff space together with a maximal atlas of local charts onto open subsets of the simplicial cone, $[0, \infty)^n \subset \mathbb{R}^n$, so that the overlap maps are homeomorphisms which preserve codimension. A manifold with corners is called *nice* if every codimension k face is contained in exactly k facets. Clearly, a nice manifold with corners is a simple convex polytope.

Let $\pi : M^{2n} \longrightarrow P^n = M^{2n}/T^n$ be the orbit map of a torus manifold and let $\mathcal{F} = \{F_1, ..., F_m\}$ be the set of facets of P^n. Denote the preimages by $M_j = \pi^{-1}(F_j)$, $1 \leq j \leq m$. Points in the relative interior of a facet F_j correspond to orbits with the same one-dimensional isotropy subgroup, which we denote by T_{F_j} . Hence M_j is a connected component of the fixed point set of the circle subgroup $T_{F_j} \subset T^n$. This implies that M_j is a T^n-invariant submanifold of codimension 2 in M, and M_j is a torus manifold over F_j with the action of the quotient torus $T^n/T_{F_j} \cong T^{n-1}$. Following the terminology of Davis and Januszkiewicz [8], we refer to M_j as the *characteristic submanifold* corresponding to the jth face $F_j \subset P^n$. The mapping $\lambda : F_j \to T_{F_j}, 1 \leq j \leq m$, is called the *characteristic function* of the torus manifold M^{2n}. Now let G be a codimension-k face of P^n and write it as an intersection of k facets: $G = F_{j_1} \cap \cdots \cap F_{j_k}$. Assign to each face G the subtorus $T_G = \prod_{F_i \supset G} T_{F_i} \subset T^{\mathcal{F}}$. Then $M_G = \pi^{-1}(G)$ is a T^n-invariant submanifold of codimension $2k$ in M, and M_G is fixed under each circle subgroup $\lambda(F_{j_p})$, $1 \leq p \leq k$.

To each n-dimensional simple convex polytope, P^n, we may associate a T^m-manifold $\mathcal{Z}_P$ with the orbit space P^n, as in [8].

Definition 2.16 (Moment Angle Manifold). *For every point $q \in P^n$ denote by $G(q)$ the unique (smallest) face containing q in its interior. For any simple polytope P^n define the* moment angle manifold

$$\mathcal{Z}_P = (T^{\mathcal{F}} \times P^n)/\sim \,=\, (T^m \times P^n)/\sim,$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T_{G(q)}$.

Note that the equivalence relation depends only on the combinatorics of P^n. In fact, this is also true for the topological and smooth type of $\mathcal{Z}_P$, that is, combinatorially equivalent simple polytopes yield homeomorphic, and, in fact, diffeomorphic, moment angle manifolds (see Proposition 4.3 in Panov [35] and the remark immediately following it).

The free action of T^m on $T^{\mathcal{F}} \times P^n$ descends to an action on $\mathcal{Z}_P$, with quotient P^n. Let $\pi_{\mathcal{Z}} : \mathcal{Z}_P \longrightarrow P^n$ be the orbit map. The action of T^m on $\mathcal{Z}_P$ is free over the interior of P^n, while each vertex $v \in P^n$ represents the orbit $\pi_{\mathcal{Z}}^{-1}(v)$ with maximal isotropy subgroup of dimension n.

In Buchstaber and Panov [3] the following facts about the space $\mathcal{Z}_P$ are proven.

Proposition 2.17. [3] *Let P^n be a combinatorial simple polytope with m facets, then*

1. *The space $\mathcal{Z}_P$ is a smooth manifold of dimension $m + n$.*
2. *If $P = P_1 \times P_2$ for some simple polytopes P_1 and P_2, then $\mathcal{Z}_P = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}$. If $G \subset P$ is a face, then $\mathcal{Z}_G$ is a submanifold of $\mathcal{Z}_P$.*

Lastly, we describe properties of the Borel construction of a locally standard torus manifold, M, with $P = M/T$ acyclic with acyclic faces. Let $M_T := ET \times_T M$ be the Borel construction, that is, the quotient of the product of M with the total space of the T-universal principal bundle ET by the diagonal action of T on both. Thus M_T is a bundle over the classifying space BT with fiber M. By Theorem 2.15, the odd degree cohomology of M vanishes. Hence the Leray-Serre spectral sequence collapses at E_2 and we obtain that $H^*(M_T) \cong H^*(BT) \otimes H^*(M)$. In particular for $n = 2$, there is a short exact sequence:

$$0 \longrightarrow H^2(BT) \longrightarrow H^2(M_T) \longrightarrow H^2(M) \longrightarrow 0. \tag{2.1}$$

2.5. **Torus Orbifolds.** In this subsection we gather some preliminary results about torus orbifolds. We first recall the definition of an orbifold. For more details about orbifolds and actions of tori on orbifolds, see Haefliger and Salem [19], and [14].

Definition 2.18 (Orbifold). *An* n-dimensional (smooth) orbifold*, denoted by* $\mathcal{O}$*, is a second-countable, Hausdorff topological space* $|\mathcal{O}|$*, called the underlying topological space of* $\mathcal{O}$*, together with an* equivalence class of n-dimensional orbifold atlases*.*

In analogy with a torus manifold, we may define a torus orbifold, as follows.

Definition 2.19 (Torus Orbifold). *A* torus orbifold*,* $\mathcal{O}$*, is a* $2n$*-dimensional, closed, orientable orbifold with an effective smooth action of an* n*-dimensional torus* T *such that* $\mathcal{O}^T \neq \emptyset$*.*

In [14], using results obtained for torus orbifolds, they prove the following theorem, which will be of use in the proof of Theorem A.

Theorem 2.20. [14] *Let* M *be an* n*-dimensional, smooth, closed, simply-connected, rationally elliptic manifold with a maximal* T^k *action. Then there is a product* $\hat{P}$ *of spheres of dimension* ≥ 3*, a torus* $\hat{T}$ *acting linearly on* $\hat{P}$*, and an effective, linear action of* T^k *on* $\hat{M} = \hat{P}/\hat{T}$*, such that there is a* T^k*-equivariant rational homotopy equivalence* $M \simeq_{\mathbb{Q}} \hat{M}$*.*

2.6. **Alexandrov Geometry.** Recall that a complete, locally compact, finite dimensional length space (X, dist) with curvature bounded from below in the triangle comparison sense is an *Alexandrov space* (see, for example, Burago, Burago and Ivanov [5]). When M is a complete, connected Riemannian manifold and G is a compact Lie group acting on M by isometries, the orbit space $X = M/G$ is equipped with the orbital distance metric induced from M, that is, the distance between $\overline{p}$ and $\overline{q}$ in X is the distance between the orbits $G(p)$ and $G(q)$ as subsets of M. If, additionally, M has sectional curvature bounded below, that is, $\sec M \geq k$, then the orbit space X is an Alexandrov space with $\mathrm{curv}\, X \geq k$.

The *space of directions* of a general Alexandrov space at a point x is by definition the completion of the space of geodesic directions at x. In the case of orbit spaces $X = M/G$, the space of directions $\Sigma_{\overline{p}}X$ at a point $\overline{p} \in X$ consists of geodesic directions and is isometric to $S_p^{\perp}/G_p$, where $S_p^{\perp}$ is the unit normal sphere to the orbit $G(p)$ at $p \in M$.

A non-empty, proper *extremal set* comprises points with spaces of directions which significantly differ from the unit round sphere. They can be defined as the sets which are "ideals" of the gradient flow of $\mathrm{dist}(p, \cdot)$ for every point p. Examples of extremal sets are isolated points with space of directions of diameter $\leq \pi/2$, the boundary of an Alexandrov space and, in a trivial sense, the entire Alexandrov space. We refer the reader to Petrunin [40] for definitions and important results.

2.7. **Geometric results in the presence of a lower curvature bound.** We now recall some general results about G-manifolds with non-negative and almost non-negative curvature which we will use throughout. Recall that a torus manifold is an example of an S^1-fixed point homogeneous manifold, indeed, of an nested S^1-fixed point homogeneous manifold. Fixed point homogeneous manifolds of positive curvature were classified in Grove and Searle [18]. More recently, the following theorem by Spindeler, [47], gives a characterization of non-negatively curved G-fixed point homogeneous manifolds .

Theorem 2.21. [47] *Assume that G acts fixed point homogeneously on a complete non-negatively curved Riemannian manifold M. Let F be a fixed point component of maximal dimension. Then there exists a smooth submanifold N of M, without boundary, such that M is diffeomorphic to the normal disk bundles $D(F)$ and $D(N)$ of F and N glued together along their common boundaries;*

$$M = D(F) \cup_\partial D(N).$$

Further, N is G-invariant and contains all singularities of M up to F.

The following two facts from [47] when M is a torus manifold of non-negative curvature, will be important for what follows.

Proposition 2.22. [47] *Let M, N and F be as in Theorem 2.21 and assume that M is a closed, simply connected torus manifold of non-negative curvature. Then N has codimension greater than or equal to 2 and F is simply-connected.*

For a non-negatively curved torus manifold, Proposition 4.5 from Wiemeler [48] shows that the quotient space, $M^{2n}/T^n = P^n$, is described as follows: P^n is a nice manifold with corners all of whose faces are acyclic and P^n is of the form

$$P^n = \prod_{i<r} \Sigma^{n_i} \times \prod_{i \geq r} \Delta^{n_i}, \tag{2.2}$$

where $\Sigma^{n_i} = S^{2n_i}/T_i^n$ and $\Delta^{n_i} = S^{2n_i+1}/T^{n_i+1}$ is an n_i-simplex. The T^{n_i}-action on S^{2n_i} is the suspension of the standard T^{n_i}-action on $\mathbb{R}^{2n_i}$ and it is easy to see that Σ^{n_i} is the suspension of Δ^{n_i-1}. Note that each n_i-simplex has n_i+1 facets and each Σ^{n_i} has n_i facets. The number of facets of P^n in this case is bounded between n and $2n$.

Using this description of the quotient space the following equivariant classification theorem is obtained in [48].

Theorem 2.23. [48] *Let M be a simply-connected, non-negatively curved torus manifold. Then M is equivariantly diffeomorphic to a quotient of a free linear torus action of*

$$\mathcal{Z}_P = \prod_{i<r} S^{2n_i} \times \prod_{i \geq r} S^{2n_i-1},\ n_i \geq 2, \tag{2.3}$$

where $\mathcal{Z}_P$ is the moment angle complex corresponding to the polytope in Display (2.2).

In the proof of Theorem 2.23, the following lemma was important. It will also be useful for the proof of Theorem A.

Lemma 2.24. [48] *Let M^{2n} be a simply-connected torus manifold with an invariant metric of non-negative curvature. Then M^{2n} is locally standard and M^{2n}/T^n and all its faces are diffeomorphic (after smoothing the corners) to standard discs D^k. Moreover, $H^{odd}(M; Z) = 0$.*

We will also make use of the following theorem from [14].

Theorem 2.25. [14] *Let M be a closed, simply-connected, non-negatively curved Riemannian manifold admitting an effective, isometric, maximal torus action. Then M is rationally elliptic.*

The following corollary of Theorem 2.25 follows by applying Theorem 5.1.

Corollary 2.26. *Let M be a closed, simply-connected, non-negatively curved Riemannian manifold admitting an effective, isometric, almost maximal torus action. Then M is rationally elliptic.*

In the proof of Theorem A, we will need to consider generalizations of Theorem 2.21, Proposition 2.22 and Lemma 2.24 to manifolds of almost non-negative curvature. We recall the definition of almost non-negative curvature here, as well as an important result of Fukaya and Yamaguchi that allows us to determine under what conditions the total space of a principal torus bundle will admit a metric of almost non-negative curvature.

Definition 2.27 (**Almost Non-Negative Curvature**). *A sequence of Riemannian manifolds* $\{(M, \mathrm{g}_\alpha)\}_{\alpha=1}^{\infty}$ *is almost non-negatively curved if there is a real number* $D > 0$ *so that*

$$\begin{aligned} \mathrm{Diam}\,(M, \mathrm{g}_\alpha) &\leq D, \\ \sec\,(M, \mathrm{g}_\alpha) &\geq -\frac{1}{\alpha}. \end{aligned}$$

A large source of examples of almost non-negatively curved manifolds can be constructed by the following result of Fukaya and Yamaguchi [11].

Theorem 2.28. [11] *Let* $F \hookrightarrow E \longrightarrow B$ *be a smooth fiber bundle with compact Lie structure group* G*, such that* B *admits a family of metrics with almost non-negative curvature and the fiber* F *admits a* G*-invariant metric of non-negative curvature. Then the total space* E *admits a family of metrics with almost non-negative curvature.*

3. The Equivariant Classification

In this section we will prove a Cross-Sectioning Theorem, which will then give us an Equivariant Classification Theorem. We will also show that when the number of facets equals the rank of the torus action, there exists a weight preserving diffeomorphism between the quotient spaces.

3.1. **Cross-Sectioning Theorem and the Equivariant Classification Theorem.** The main tool for an equivariant diffeomorphism classification is giving by the existence of smooth cross-sections as proven in the following theorem.

Theorem 3.1. *Let* T^k *act smoothly on a smooth, closed,* n*-dimensional manifold,* M^n *such that* $M^* = M^n/T^k$ *is an* $(n-k)$*-dimensional disk,* D^{n-k}*. Suppose further that all interior points of* M^* *are principal orbits and that points on the boundary,* ∂M^**, correspond to singular orbits with connected isotropy subgroups. Then the orbit map* $\pi : M^n \to M^*$ *has a cross-section.*

The proof is a direct generalization of the equivariant classification theorem for T^3-actions on M^6 in [28]. Similar techniques are also used in work of Raymond [41] and Orlik and Raymond [42, 43, 44]. The main topological tool in the proof comes from obstruction theory, that is, the obstruction to extending a map $A \longrightarrow X$ to a map $W \longrightarrow X$ where X is a connected CW-complex and (W, A) is a CW-pair. Such an extension always exists, that is the obstruction vanishes, if $H^{n+1}(W, A, \pi_n(X)) = 0$ for all n. For more details on obstruction theory see, for example, Davis and Kirk [9].

In the following two lemmas, required for the proof of Theorem 3.1, we will be considering a closed $T^k = T_1 \times \cdots \times T_k$-invariant subset C of the closed manifold, M, with M as in Theorem 3.1. We choose C so that its orbit space under the torus action, $C^* \subset M^* = D^{n-k}$ is a closed conical section of M^*, with conical orbit space, as in Definition 2.11. That is, $C^* \cong D^{n-k} = \overline{C(D^{n-k-1})}$, is a closed cone over D^{n-k-1}. Moreover, we choose C so that its intersection with the boundary, ∂M^*, is homeomorphic to D^{n-k-1}. In Figure 3.1, we illustrate this orbit space C^* and its homeomorphic image D^{n-k}. We denote the $(n-k-1)$-disk which we cone over by S^- and the cone point, or vertex, will be denoted by $p^* \in S^+$, where $S^+ = C^* \cap \partial M^*$.

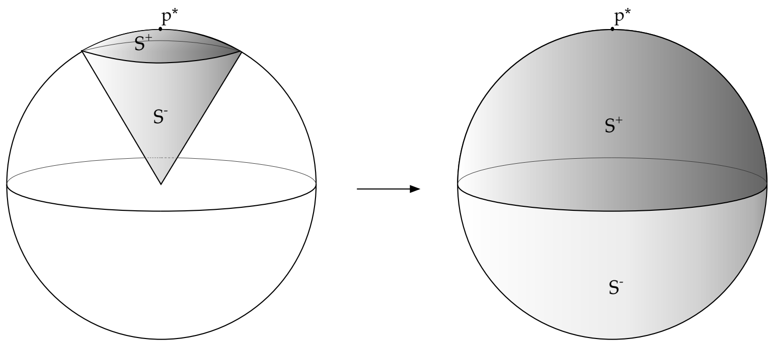

FIGURE 3.1. The conical section $C^* \subset M^*$ and its homeomorphic image D^{n-k}.

Since isotropies are constant along rays from p^*, we can partition (S^+, p^*) into cells of dimension $(n-k-1)$, denoted by $U_{k-i+1}, \ldots, U_k$, provided they all intersect in p^*, that is, $p^* \in \bigcap_{j=k-i+1}^{i} U_j$. Note that to each U_l we associate the corresponding circle isotropy subgroup T_l, where $k-i+1 \leq l \leq k$. By assumption, the T_l generate T^i and each pair of distinct circles has trivial intersection, that is, $T_{k-i+1} \times \cdots \times T_k = T^i$.

This gives us a weighted decomposition of (S^+, p^*), which we denote by

$$\{(U_{k-i+1}, T_{k-i+1}), \ldots, (U_k, T_k)\}.$$

It is understood then in this decomposition that each intersection of j-cells in a $(j-1)$-cell corresponds to the connected isotropy subgroup of the T^k-action generated by the isotropy subgroups associated to each of the j-cells.

The simplest possible decomposition is as in Lemma 3.2, where the decomposition of (S^+, p^*) is given as $\{(U, T_k)\}$ and is illustrated in the right hand figure of Figure 3.1. The next simplest is given as $\{(U_{k-1}, T_{k-1})(U_k, T_k)\}$ and is illustrated in Figure 3.2. The most general decomposition will be the one where p^* corresponds to an orbit with T^k isotropy, and whose weighted decomposition is $\{(U_{k-i+1}, T_{k-i+1}), \ldots, (U_k, T_k)\}$. Note that by construction all non-trivial isotropies are connected and correspond to points on S^+, all other orbits are principal.

In the following lemma, we begin with the simplest case and show that we can construct a cross-section for C^*.

Lemma 3.2. *Let $T^k = T_1 \times \cdots \times T_k$ act smoothly on a smooth, closed n-dimensional subspace $C \subset M$, where M is a smooth, closed n-dimensional closed manifold, and C has quotient space, C^*, as described above. Suppose $(S^+, p^*) = \{(U_k, T_k)\}$. Then there exists a cross-section. Further, suppose a cross-section is given on an $(n-k-1)$-cell $A \subseteq S^-$. Then it can be extended to all of D^{n-k}.*

Proof of Lemma 3.2. Recall that the orbit space is a closed cone with vertex p^*, the north pole of D^{n-k}. Since the orbit structure is conical and $G_p = T_k$, by Theorem 2.12, C is equivariantly homeomorphic to $T^{k-1} \times D^{n-k+1}$ with T_k acting orthogonally on D^{n-k+1}. Construct a cross section from D^{n-k} to D^{n-k+1} using the "inverse" of the projection map given via the orthogonal action of T_k on D^{n-k+1} and then we construct a section from D^{n-k+1} to $T^{k-1} \times D^{n-k+1}$ by sending an arbitrary point $x \in D^{n-k+1}$ to $(t, x) \in T^{k-1} \times D^{n-k+1}$ for some $t \in T^{k-1}$.

Now suppose a cross-section s is given on a $(n-k-1)$-cell $A \subseteq S^-$. Let $A' = A \cap (D^{n-k} \setminus S^+)$ and $\pi : C \longrightarrow C^* \cong D^{n-k}$ the orbit map. Then $\pi^{-1}(D^{n-k} \setminus S^+)$ is a principal T^{n-k}-bundle over $(D^{n-k} \setminus S^+)$. Using the long exact sequence for relative cohomology and

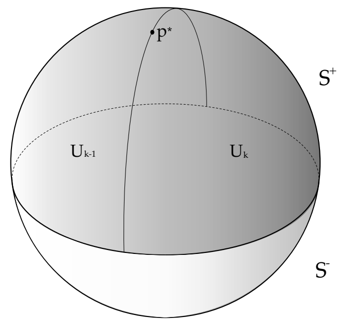

FIGURE 3.2. The upper hemisphere $S^+ = U_{k-1} \cup U_k$.

excision, it follows that $H^i((D^{n-k} \setminus S^+), A') = 0$ for all $i > 0$. Thus, by obstruction theory, we may assume that s is defined on $(D^{n-k} \setminus S^+) \cup A$.

We now obtain the following diagram, where $\pi_1 : C \to \bar{C}_1 = C/T^{k-1}$ and $\pi_2 : \bar{C}_1 \to C^* \cong D^{n-k}$.

$$\begin{tikzcd}
C \arrow[r, "\pi_1"] \arrow[d, "\pi"] & \bar{C}_1 \arrow[l, bend right, "s_1"'] \arrow[dl, "\pi_2"] \\
D^{n-k} \arrow[u, bend left, "s"] \arrow[ur, bend right, "s_2"'] &
\end{tikzcd}$$

Let $s_2 = \pi_1 \circ s$. Then s_2 is a cross-section to π_2 defined on $(D^{n-k} \setminus S^+) \cup A$. But S^+ corresponds to the set of fixed points of the T_k action on $\bar{C}_1$, so we can define s_2 on all of D^{n-k}. In order to show continuity of s_2 we first describe D^{n-k} as $I \times S^+$, where I is an interval. Note that $\pi_2^{-1}(S^+) \cong D^{n-k-1}$ and $\pi_2^{-1}(I) \cong \bar{C}(T_k) \cong D^2$ and the T_k-action on $\pi_2^{-1}(I)$ is rotation. Hence the T_k action on $C_1 \cong D^{n-k+1} \cong D^2 \times D^{n-k-1}$ is rotation on the first factor and trivial on the second. An orbit can be described as $\{re^{i\theta}, Re^{i\Theta} | 0 \leq \theta < 2\pi, 0 \leq \Theta < 2\pi\}$ where

$$Re^{i\Theta} = \begin{cases} (r_1 e^{i\theta_1}, \ldots, r_{\frac{n-k-1}{2}} e^{i\theta_{\frac{n-k-1}{2}}}) & \text{if } n-k-1 \text{ is even,} \\ (r_1 e^{i\theta_1}, \ldots, r_{\frac{n-k-2}{2}} e^{i\theta_{\frac{n-k-2}{2}}}, 1) & \text{if } n-k-1 \text{ is odd.} \end{cases}$$

Note that the fixed point set of T_k on $D^2 \times D^{n-k-1}$ is $\{0\} \times D^{n-k-1}$. Now let $q = (0, Se^{i\Sigma}) \in \mathrm{Fix}(T_k, D^2 \times D^{n-k-1})$ and let $\{q_n^*\} = \{(r_n e^{i\theta_n}, R_n e^{i\Theta_n})^*\}$ be a sequence in M^* converging to q^*. Then $r_n \longrightarrow 0, R_n \longrightarrow S$ and $\Theta_n \longrightarrow \Sigma$. But then the sequence $\{s_2(q_n^*)\}$ will be of the form $\{r_n e^{i\theta_n}, R_n e^{i\Theta_n}\}$ which converges to $q = (0, Se^{i\Sigma})$. Hence s_2 is continuous.

Next we need a cross-section s_1 defined on $s_2(C^*) \cong D^{n-k}$. Let $s_1 = s \circ \pi_2 : s_2(C^*) \setminus s_2(S^+ \setminus A) \longrightarrow C$. Now $\pi_1^{-1}(s_2(C^*))$ is a principal T^{k-1}-bundle over $s_2(C^*) \cong D^{n-k}$ and we obtain that s_1 is in fact a cross-section of π_1 defined on $s_2(C^*) \setminus s_2(S^+ \setminus A)$. Since $s_2(S^+ \setminus A)$ is a homology $(n-k-1)$-cell on the boundary of $s_2(C^*)$, it follows that $H^i(s_2(C^*), s_2(C^*) \setminus s_2(S^+ \setminus A)) = 0$ for all $i > 0$. Again by obstruction theory this implies that s_1 can be extended to all of $s_2(C^*)$. Thus $s_1 \circ s_2$ extends s to all of C^*.

□

We are now ready to construct a cross-section for a general decomposition of C^*.

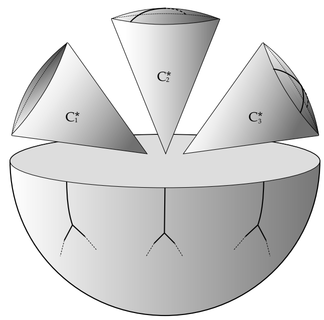

FIGURE 3.3. A partial decomposition of the quotient space, M^*.

Lemma 3.3. *Let $T^k = T_1 \times \cdots \times T_k$ act smoothly on a smooth, closed n-dimensional subspace $C \subset M$, where M is a smooth, closed n-dimensional closed manifold, and C has quotient space, C^*, as described above. Let the decomposition of (S^+, p^*) be given by $\{(U_{k-i+1}, T_{k-i+1}), \dots, (U_k, T_k)\}$, with $1 < i \leq n-k \leq k$. Then there exists a cross-section. Further, suppose a cross-section is given on an $(n-k-1)$-cell $A \subseteq S^-$. Then it can be extended to all of D^{n-k}.*

We leave the proof of Lemma 3.3 to the reader, as it is a straightforward generalization of the proof of Lemma 3.2. We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. As stated earlier, the proof is a direct generalization of the equivariant classification theorem for T^3-actions on M^6 in [28], and which we include for the sake of completeness. First decompose the orbit space $M^* \cong D^{n-k}$ into a collection of conical sections $\{C_i^*\}_{i=1}^m$, with $C_i^* \cong D^{n-k}$ for each $i \in \{1, \dots, m\}$, and such that the orbit structure for each $C_i \subset M^n$ is conical as in Definition 2.11. As we saw in the proof of Lemma 3.3, each $C_i \cong T^{k-l} \times D^{n-k+l}$ for some $l \in \{1, \dots k\}$ and therefore a cross-section exists on each C_i^*. An obstruction theory argument shows that a cross-section given on an $(n-k-1)$-cell $A \subset S^- \subset C_i^*$ can be extended to all of C_i^*.

In order to create a cross-section on all of M^*, we start by defining a cross-section s on C_1^*. We then attach C_2^* to C_1^* along an $(n-k-1)$-cell A and so we have a cross section s defined on C_1^* and on $A \subset S^- \subset C_2^*$. By Lemma 3.3, we can then extend the cross-section s to all of C_2*. Continuing this process we can extend s to all of M^*. Figure 3.3 illustrates this process for a partial decomposition of M^*. □

It is clear that if M_1^n is equivariantly diffeomorphic to M_2^n then the corresponding orbit spaces will be weight-preserving diffeomorphic. Vice versa, one can use the weight-preserving diffeomorphism and the existence of a cross-section to construct a G-equivariant diffeomorphism between M_1^n and M_2^n. Note that smoothness of the action implies that that orbit map is smooth, and hence we may choose a smooth cross-section and obtain the following equivariant diffeomorphism classification.

Theorem 3.4. *Let G act smoothly on smooth, closed, n-dimensional manifolds, M_1^n and M_2^n and suppose that the orbit maps $\pi : M_i^n \to M^* = M_i^n/G$ have cross-sections for $i = 1, 2$. Then there exists an equivariant diffeomorphism h from M_1^n onto M_2^n if and only if there exists a weight-preserving diffeomorphism h^* from M_1^* onto M_2^*. Furthermore, if M_1 and M_2 are oriented and the orientations of M_1^* and M_2^* are those induced by M_1 and M_2, then h is orientation preserving if and only if h^* is orientation preserving.*

3.2. **Creating a weight preserving diffeomorphism.** In this subsection we will consider the following special case of Theorem A. We assume that the number of facets of the quotient space of M^n, under a maximal T^k action, is equal to the rank of the T^k-action. We will show in Proposition 3.11 that under these hypotheses, there exists a manifold, $\bar{M}^n$, equivariantly diffeomorphic to M^n, admitting an isometric, effective and maximal T^k action for which the free dimension is equal to the free rank.

Lemma 3.5. *Let T^k act isometrically, effectively and maximally on a closed, simply-connected, n-dimensional Riemannian manifold, whose quotient space M/T is*

$$P^{n-k} = \prod_{i<r} \Sigma^{n_i} \times \prod_{i\geq r} \Delta^{n_i},$$

and the number of facets of P^{n-k} is equal to k. Suppose that all singular isotropies are connected and correspond to points on the boundary and all other orbits are principal. Then there exists a weight preserving diffeomorphism $\phi : P^{n-k} \to P^{n-k}$ taking the weight vectors, $\{\mathbf{a}_i\}$, of P^{n-k} to $\{\phi(\mathbf{a}_i)\}$, where $\phi(\mathbf{a}_i)$ is the unit vector with the value ± 1 in the i-th position.

Proof. By Corollary 2.9, it follows that the circle isotropy subgroups generate T^k. So we may assign a weight to each facet, corresponding to the circle isotropy subgroup of T^k. Each weight may be written as a k-vector and we may group all the weights in a $k \times k$ matrix. By Proposition 2.10, the determinant of this matrix is equal to ± 1. Since the $k \times k$ matrix of the weights has integer entries and non-zero determinant, using the Smith normal form, it can always be diagonalized. After diagonalizing, we obtain a $k \times k$ matrix with ± 1 entries on the diagonal, as desired. □

Lemma 3.6. *With the same hypotheses as in Lemma 3.5, suppose that the matrix of weights of P^{n-k} is the matrix with ± 1 entries along the diagonal. Then the T^k action has $2k - n$ freely acting circles.*

Proof. By assumption the matrix of weights of P^{n-k} is the matrix with ± 1 entries along the diagonal. Hence all circle isotropy subgroups are mutually orthogonal. In order to show that the T^k action has $(2k - n)$ freely acting circles, we claim that it suffices to find $(2k - n)$ pairs of opposing faces on the polytope. For each such pair of opposing faces, we consider the diagonal circle in the subgroup of T^k generated by the corresponding isotropy subgroups of each opposing face. Since the faces are opposing, they will not intersect in any lower-dimensional face and hence the diagonal circle in this subgroup will intersect all isotropy subgroups trivially and corresponds to a circle subgroup of T^k that acts freely.

Recall that Σ^{n_i} has n_i facets and Δ^{n_i} has $n_i + 1$ facets and that the number of facets in a product of polytopes equals the sum of the facets in each polytope. It is clear that Σ^{n_i}, since it is the suspension of a Δ^{n_i-1}, has no pairs of opposing faces that do not intersect in a lower dimensional face. Further, in order to have the correct free rank, P^{n-k} must contain the product of exactly $(2k - n)$ simplices of dimension n_i, Δ^{n_i}. Thus, in order to find the $(2k - n)$ pairs of opposing faces, it suffices to first show that for any product of $(2k - n)$ Δ^{n_i}, that we can find $(2k - n)$ opposing faces. The product of these pairs of opposing faces with the remaining product of Σ^{n_i}s in P^{n-k} will then form the desired set of $(2k - n)$ pairs of opposing faces.

We will use barycentric coordinates for the n-simplex, $\Delta^n = \sum_{i=0}^n s_i\, v_i$ with vertices $v_0, \ldots, v_n$ where $(s_0, \ldots s_n) \in \mathbb{R}^{n+1}$, $\sum_{i=0}^n s_i = 1$ and $s_i \geq 0$ for $i = 0, \ldots n$. We define the set

$$\Delta_i^{n-1} = \{(s_0, \ldots, s_{i-1}, 0, s_i, \ldots s_n) | \sum_{i=0}^n s_i = 1,\ s_i \geq 0 \text{ for } i = 0, \ldots n\}$$

corresponding to the $(n-1)$-simplex to be *opposite* the vertex

$$v_i = (0, \dots 0, 1, 0, \dots 0)$$

with 1 in the i-th coordinate.

Consider the following canonical example:

Example 3.7. *Consider the simplex, Δ^k, with $k+1$ facets, arising as the quotient of a T^{k+1} action on M^{2k+1}, with free rank equal to 1. Consider the pair of opposing faces given by*

$$\{\Delta_i^{k-1}, v_i\}.$$

Let T_i^1 be the isotropy subgroup corresponding to Δ_i^{k-1} and let T_i^k be the isotropy subgroup corresponding to v_i. Then it is clear that $T_i^1 \cap T_i^k$ intersect trivially in T^{k+1}. The diagonal circle in T^{k+1} generated by T_i^1 and T_i^k intersects all isotropy groups trivially and hence acts freely.

Example 3.8. *Consider the simplex, $\Delta^{k-l} \times \Delta^l$, with $k+2$ facets, arising as the quotient of a T^{k+2} action on M^{2k+2}, with free rank equal to 2. Consider the two pairs of opposing faces given by*

$$\{\Delta_i^{k-l-1} \times \Delta^l, v_i \times \Delta^l\} \text{ and } \{\Delta^{k-l} \times \Delta_j^{l-1}, \Delta^{k-l} \times v_j\}.$$

Let T_i^1 be the isotropy subgroup corresponding to $\Delta_i^{k-l-1} \times \Delta^l$ and let T_i^{k-l} be the isotropy subgroup corresponding to $v_i \times \Delta^l$ and T_j^1 be the isotropy subgroup corresponding to $\Delta^{k-l} \times \Delta_j^{l-1}$ and let T_j^l be the isotropy subgroup corresponding to $\Delta^{k-l} \times v_j$. Then it is clear that $T_i^1 \cap T_i^{k-l}$ intersect trivially in T^{k+2} and so do T_j^1 and T_j^l. The diagonal circles in $< T_i^1, T_i^{k-l} >= T^{k-l+1}$ and in $< T_j^1, T_j^l >= T^{l+1}$ intersects all isotropy groups trivially and hence act freely.

For the sake of simplicity of notation, we will prove only the case when $P^{n-k} = \prod_{i=1}^{2k-n} \Delta^{n_i}$. We will proceed by induction on the number of simplices contained in the product. The base case is covered by Example 3.7. Let $\prod_\Delta^l = \prod_{i=1}^l \Delta^{n_i}$ with $\sum_{i=1}^l n_i + l = k$ facets arising as the quotient of a T^k-action on M^n, with free rank equal to $2k-n = l$, and assume by the induction hypothesis that on any subproduct $\prod_\Delta^{l-1} \subset \prod_\Delta^l$ we can find $(l-1)$ pairs of opposing faces.

Without loss of generality, we let $\prod_\Delta^l = \prod_\Delta^{l-1} \times \Delta^{n_l}$. We denote $\Delta^{n_1} \times \cdots \times \widehat{\Delta^{n_j}} \times \Delta^{n_l}$ by $\prod_\Delta^{l,j}$. Then we get l freely acting circles from opposing faces given by $\{\prod_\Delta^{l,j} \times \Delta_i^{n_j-1}, \prod_\Delta^{l,j} \times v_i\}$ for $j = 1, \dots, l-1$ and one additional one by the construction in Example 3.8 with the pair of opposing faces given by $\{\prod_\Delta^{l-1} \times \Delta_i^{n_l-1}, \prod_\Delta^{l-1} \times v_i\}$. Hence we get l pairs of opposing faces, as desired and the lemma is proved.

□

We now establish some preliminary facts about the quotient space $P^{n-k} = M^n/T^k$, when M^n is a closed, non-negatively curved manifold and the T^k action is maximal. By work in [14], it follows that P^{n-k}, is a polytope of the form as in Display (2.2), with corresponding moment angle manifold equal to a product of spheres of dimension greater than or equal to three, as in Display (2.3).

The following Lemma gives us information about the structure of the quotient space, P^{n-k} as well as a complete description of the corresponding isotropy groups.

Lemma 3.9. *Let T^k act isometrically, effectively and maximally on M^n, a closed, non-negatively curved Riemannian manifold. Then the following hold for the quotient space $P = M/T$:*

(1) *All isotropy subgroups corresponding to boundary points on P are connected; and*
(2) *All boundary points correspond to singular orbits and all interior points correspond to principal orbits.*

Proof. Since every face of P contains a vertex, the maximality of the action implies that all isotropy subgroups on the boundary of P are connected, giving us Part (1).

The image of any point of singular or exceptional isotropy will be contained in an extremal set of P. Let $F \subset M$ correspond to the inverse image of any facet $\bar{F}$ of P. The points of $\bar{F}$ correspond to orbits with circle isotropy. By concavity of the distance function $\mathrm{dist}_{\bar{F}}(\cdot)$, it follows that any point belonging to an extremal set must lie at maximal distance from $\bar{F}$ and therefore must lie on the boundary of P. So the interior of P must consist entirely of principal orbits, giving us Part (2). □

The following lemma then follows by a direct application of Theorem 3.1.

Lemma 3.10. *The map $\pi : \mathcal{Z}_{\phi(P^{n-k})} \to \phi(P^{n-k})$ admits a cross-section.*

By Lemmas 3.5, 3.6, and 3.10 and the Equivariant Classification Theorem 3.4, we obtain the following Proposition 3.11.

Proposition 3.11. *Let T^k act isometrically and effectively on M^n, a closed, simply-connected, n-dimensional Riemannian manifold with free rank equal to $2k - n$, whose quotient space M/T is*

$$P^{n-k} = \prod_{i<r} \Sigma^{n_i} \times \prod_{i \geq r} \Delta^{n_i},$$

and the number of facets of P^{n-k} is equal to k. Suppose that all singular isotropies are connected and correspond to points on the boundary and all other orbits are principal. Then there exists an equivariant diffeomorphism between M and $\bar{M}$, where $\bar{M} = \mathcal{Z}_{\phi(P^{n-k})}$ admits a freely acting torus of rank $2k-n$ and ϕ is the weight-preserving diffeomorphism of Lemma 3.5.

Observe that if we strengthen the hypotheses of Proposition 3.11 to make the torus action maximal and assume that M^n is non-negatively curved, we can then drop the hypotheses on the isotropy groups, as Lemma 3.9 guarantees that they will be satisfied.

4. Almost non-negative curvature and locally standard actions

In this section, using a generalization of work of [47], we extend a result of [48] that allows us to decide when a torus manifold with non-negatively curved quotient space is locally standard.

It follows from the proof of Theorem 2.21, in fact, that we only need assume that M is complete and the quotient M/G is a non-negatively curved Alexandrov space to obtain the same result, so we reformulate the theorem as follows.

Theorem 4.1. *Assume that G acts fixed point homogeneously on a complete Riemannian manifold M such that M/G is a non-negatively curved Alexandrov space. Let F be a fixed point component of maximal dimension. Then there exists a smooth submanifold N of M, without boundary, such that M is diffeomorphic to the normal disk bundles $D(F)$ and $D(N)$ of F and N glued together along their common boundaries;*

$$M = D(F) \cup_{\partial} D(N).$$

Further, N is G-invariant and contains all singularities of M up to F.

It also follows from the proof of Proposition 2.22, that we may weaken the hypotheses to assume only that M is complete and the quotient M/G is a non-negatively curved Alexandrov space as follows.

Proposition 4.2. *Let M, N and F be as in Theorem 4.1 and assume that M is a closed, simply connected torus manifold. Then N has codimension greater than or equal to 2 and F is simply-connected.*

Remark 4.3. *Using Theorem B of Searle and Wilhelm* [46]*, which allows one to lift a metric of almost non-negative curvature on a G-manifold M, provided M/G is almost non-negatively curved, we note that M as above admits a G-invariant family of metrics of almost non-negative curvature. So Theorem 4.1 and Proposition 4.2 hold for the class of G-invariant almost non-negatively curved manifolds with non-negatively curved quotient spaces.*

We now introduce the following condition on the components of fixed point sets of almost non-negatively curved manifolds with non-negatively curved quotient space.

Condition F. *Let T^k act on M^n, a closed, simply-connected, manifold admitting a family of metrics of almost non-negative curvature and assume that M^n/T^k is a non-negatively curved Alexandrov space. Then for any component $F \subset M^{T^i}$, $1 \leq i \leq k$, the quotient F/T^k is a non-negatively curved Alexandrov space.*

We can then prove the following generalization of Lemma 6.3 of [48], for non-negatively curved torus manifolds.

Theorem 4.4. *Let M be a closed, simply-connected, torus manifold admitting a family of metrics of almost non-negative curvature and assume that M/T is a non-negatively curved Alexandrov space. Suppose that the manifold satisfies Condition F. Then M is locally standard and M/T and all its faces are diffeomorphic (after smoothing the corners) to standard disks. In particular, $H^{odd}(M) = 0$.*

We leave the proof of Theorem 4.4 to the reader as it follows along the same lines as the proof of Lemma 6.3 in [48], using induction on the dimension of M.

One can also generalize Theorem 2.23 for the class of manifolds considered in Theorem 4.4. The only element in the proof of Theorem 2.23, that we have not already generalized to this class of manifolds and which is curvature dependent is Lemma 4.2 of [48]. However, it is clear that Lemma 4.2 of [48] does hold, since we require the quotient space to be non-negatively curved for this class of manifolds, and thus, one can still apply Lemma 4.1 in [16] to obtain the result. Thus, combining our Theorems 4.1 and 4.4, along with Lemmas 6.4, 6.7, 6.8 in [48] and an argument from the proof of Theorem 4.1 in [48], yields the following theorem.

Theorem 4.5. *Let M be a closed, simply-connected, torus manifold admitting a family of metrics of almost non-negative curvature and assume that M/T is a non-negatively curved Alexandrov space. Suppose that the manifold satisfies Condition F. Then M is the equivariantly diffeomorphic to the quotient by a free, linear torus action on a product of spheres of dimensions greater than or equal to three.*

5. Almost maximal is maximal and a general lower bound for free rank

In this section we will establish two important facts about torus actions. We will first show that for closed, simply-connected, non-negatively curved manifolds an almost maximal action is actually maximal and then we will find a lower bound for the free rank of an isometric, effective torus action on a closed Alexandrov with a lower curvature bound.

5.1. **Almost Maximal Is Maximal.** In Theorem 5.1 below, we will now establish on closed, simply-connected, non-negatively curved manifolds that an almost maximal action is actually maximal. As mentioned in the Introduction, having proven Theorem 5.1, it then follows that in order to prove Theorem A it suffices to consider the case where the action is maximal.

Theorem 5.1. *Let T^k act isometrically, effectively and almost maximally on M^n, a simply-connected, closed, non-negatively curved Riemannian n-manifold, with $k \geq \lfloor (n+1)/2 \rfloor$. Then the action is maximal.*

Before we begin the proof of Theorem 5.1, we first need the following Lemma.

Lemma 5.2. *Let M be a manifold with rk$(H_1(M;\mathbb{Z})) = k$, $k \in \mathbb{Z}^+$. If M admits a disk bundle decomposition*

$$M = D(N_1) \cup_E D(N_2),$$

where N_1, N_2 are smooth submanifolds of M and N_1 is orientable and of codimension two, then both rk$(H_1(N_1;\mathbb{Z}))$ and rk$(H_1(N_2;\mathbb{Z}))$ are less than or equal to $k+1$.

Here $\mathrm{rk}(G)$ denotes the number of $\mathbb{Z}$ factors in the finitely generated abelian group G.

Proof. It follows from the Mayer Vietoris sequence of the decomposition and the simple-connectivity of M that the following sequence is exact

$$H_1(E) \to H_1(N_1) \oplus H_1(N_2) \to \mathbb{Z}^k \oplus \Gamma \to 0, \tag{5.1}$$

where Γ is a finite abelian group. Since E is a circle bundle over N_1, it follows from the Gysin sequence for homology that $\mathrm{rk}(H_1(E)) \leq 1+\mathrm{rk}(H_1(N_1))$. The same statement follows for $\mathrm{rk}(H_1(N_2))$ since E is either a circle or sphere bundle over N_2. Using these facts and the exactness of the sequence in Display (5.1), it follows that

$$\mathrm{rk}(H_1(N_1)) + \mathrm{rk}(H_1(N_2)) \leq \mathrm{rk}(H_1(E)) \leq \mathrm{rk}(H_1(N_1)) + 1,$$

$$\mathrm{rk}(H_1(N_1)) + \mathrm{rk}(H_1(N_2)) \leq \mathrm{rk}(H_1(E)) \leq \mathrm{rk}(H_1(N_2)) + 1,$$

and the lemma is proven. □

Proof of Theorem 5.1. Suppose that there is an orbit of dimension $2k-n+1 = m$. The isotropy subgroup of this orbit is of rank $k-m = n-k-1$. Consider the action of the T^{n-k-1} on the unit normal $S^{2n-2k-2}$. Observe that the isotropy action is maximal and of maximal symmetry rank. Hence there is a codimension two submanifold, N^{n-2}, of M^n fixed by some circle subgroup of the isotropy subgroup T^{n-k-1} and there is an induced action of T^{k-1} on N^{n-2}, of cohomogeneity $n-k-1 = k-m$. That is, M^n is S^1-fixed point homogeneous. In fact, it is nested S^1-fixed point homogeneous as there is a nested tower of fixed point sets containing the smallest orbit T^m, as follows:

$$T^m \subset N^{m+1} \subset \cdots \subset N^{n-2} \subset M^n,$$

and T^m and N^{m+1} are both fixed by T^{k-m}. Note that since M^n is nested S^1-fixed point homogeneous, each N^l is a non-negatively curved S^1-fixed point homogeneous manifold. So it follows by Theorem 2.21 that each N^l admits a disk bundle decomposition as in the statement of Lemma 5.2.

The induced action of T^m on N^{m+1} is by cohomogeneity one. We claim that the action must have circle isotropy. If it does not, then by the classification of cohomogeneity one torus actions (see Mostert [31], Neumann [32], Pak [34], Parker [36]), N^{m+1} is equivariantly diffeomorphic to T^{m+1} and has first integer homology group $\mathbb{Z}^{m+1}$. However, applying

Lemma 5.2 m times, we see that the number of $\mathbb{Z}$ factors in $H_1(N^{m+1};\mathbb{Z})$ must be less than or equal to m.

Thus there is an orbit with T^{k-m+1} isotropy and the smallest dimensional orbit must have dimension $m-1$, a contradiction. □

5.2. **Lower bounds for the free rank.** In order to establish the lower bound for the free rank, we first need the following proposition, which establishes the existence of a T^k fixed point when the torus actions has no circle subgroup acting almost freely.

Proposition 5.3. *Let T^k act isometrically on X^n, a closed n-dimensional Alexandrov space with a lower curvature bound. Suppose that every circle subgroup has a nonempty fixed point set, or equivalently, no circle subgroup acts almost freely. Then T^k has a fixed point.*

The proof is a slight modification of the proof of Lemma 3.8 of Harvey and Searle [20]. Since it is short, we repeat it here for the sake of completeness.

Proof. Consider an infinite cyclic subgroup $G = \langle g \rangle$ of a dense 1-parameter subgroup of T^k. Then G fixes some point $x_0 \in X$ since the dense 1-parameter subgroup does by hypothesis. Now consider a sequence of infinite cyclic subgroups G_i such that the distance between the generators g_i and the identity element of the torus converges to 0. The corresponding sequence of fixed points $x_i \in X^n$ then converges to a fixed point $x \in X$ of T^k. □

The following proposition establishes a lower bound of the free rank of a general torus action.

Proposition 5.4. *Let T^k act isometrically and effectively on X^n, a closed Alexandrov space with a lower curvature bound, with $k \geq \lfloor (n+1)/2 \rfloor$. Then the free rank of the T^k action is greater than or equal to $2k-n$.*

Proof. Let $T^l \subset T^k$ be the largest subtorus that acts almost freely, and suppose that $l < 2k-n$. Since $l < 2k-n$, it follows that

$$k - l > \frac{n-l}{2} \geq \lfloor \frac{n-l}{2} \rfloor.$$

Now $T^{k-l} \cong T^k/T^l$ acts on $X^{n-l} = X^n/T^l$, a closed Alexandrov space with the same lower curvature bound, and no circle subgroup of T^{k-l} acts almost freely on X^{n-l}. By Lemma 5.3, there is a point $\bar{p} \in X^{n-l}$ fixed by T^{k-l}. So, there is an action of T^{k-l} on $\Sigma_{\bar{p}}$, the unit normal space of directions to this orbit, which is itself a closed Alexandrov space of dimension $n-l-1$ with curvature bounded below by 1. The Maximal Symmetry Rank Theorem for positively curved Alexandrov spaces in [20] states that for a rank j torus-action on X^m, as is also true in the manifold case, $j \leq \lfloor (m+1)/2 \rfloor$. We then have

$$k - l \leq \lfloor \frac{n-l}{2} \rfloor,$$

a contradiction. Hence the bound holds. □

The following corollary follows directly from Propositions 5.3 and 5.4 and Theorem 5.1 combined with Corollary 6.3 of Ch. II of Bredon [2].

Corollary 5.5. *Let T^k act isometrically and effectively on M^n, a closed, simply-connected, non-negatively curved Riemannian manifold, with $k \geq \lfloor (n+1)/2 \rfloor$. Suppose that the free rank of the action is less than or equal to $2k-n+1$. Then, the following hold:*

(1) *If the free dimension is equal to the free rank, then the quotient space, $M^{2n-2k} = M^n/T^{2k-n}$, admits a T^{n-k} action and is a closed, simply-connected, non-negatively curved torus manifold.*

(2) *If the free dimension is not equal to the free rank, then the quotient space, $X^{2n-2k} = M^n/T^{2k-n}$, admits a T^{n-k} action and is a closed, simply-connected, non-negatively curved (in the Alexandrov sense) torus orbifold.*

6. The Proof of Theorem A

We first give a brief outline of the proof of Theorem A. We have shown in Theorem 5.1 that an almost maximal action is actually maximal for this class of manifolds, so we only need to prove Theorem A for maximal actions, that is, when the free rank is equal to $2k-n$.

To prove the equivariant classification we split the proof into two cases: Case (1), where the free dimension equals the free rank, and Case (2), where the subtorus corresponding to the free rank acts almost freely but not freely. In both Case (1) and Case (2) there are two further subcases: Subcase (a), where the rank of the action is equal to the number of facets of the orbit space M^n/T^k and Subcase (b), where the rank of the action is strictly less than the number of facets of the orbit space M^n/T^k.

In Cases (1a) and (1b), we show that the quotient of M^n by the freely acting subtorus, M^n/T^{2k-n}, is a torus manifold of non-negative curvature and the result will follow in each case by Corollary 6.2 and Corollary 6.4, respectively. For Case (2), the quotient of M^n is a torus orbifold of non-negative curvature. In Case (2a), we use Proposition 3.11 to show that M^n is equivariantly diffeomorphic to an n-dimensional manifold $\bar{M}$ which admits a freely acting torus of rank $2k-n$, such that $\bar{M}/T^{2k-n}$ is a torus manifold admitting a family of metrics with almost non-negative curvature and the quotient $\bar{M}/T^k$ is weight-preserving diffeomorphic to M/T^k. By Lemma 2.24, we see that the torus manifold $\bar{M}/T^{2k-n}$ is locally standard and then we can once again use Theorem 6.2 to obtain the result.

For Case (2b), one can show that there is a closed, principal T^l bundle over M^n, with simply-connected total space, N^{n+l}, admitting a T^{l+k}-invariant family of metrics of almost non-negative curvature and such that the rank of the torus action on N is equal to the number of facets of the orbit space $M^n/T^k = N^{n+l}/T^{k+l}$. We then appeal to Proposition 3.11, as in Case (2a), to find $\bar{N}$ such that $\bar{N}/T^{2k+l-n}$ is a torus manifold admitting a family of metrics of almost non-negative curvature and proceed in a similar fashion as in Case (2a).

6.1. Proof of Case (1) of Theorem A.

Recall that we assume here that the free dimension of the torus action is equal to the free rank. As detailed above, we will break the proof into two cases as follows.

6.1.1. Case (1a): The number of facets of M/T is equal to the rank of the torus action.

Part (1) of Corollary 5.5 tells us that the quotient of M by the free action is a torus manifold, that is, M is a principal torus bundle over a torus manifold. We will show in this subsection that M is equivariantly diffeomorphic to the corresponding moment angle manifold of the simple polytope $P = M^{2n-2k}/T^{n-k} = M^n/T^k$.

We assume that the manifold N is of dimension $2n+p$, admits a T^{n+p} action and is a T^p principal bundle over the torus manifold M^{2n}. We will first show in Theorem 6.1 that if the number of facets of the simple polytope $P^n = N^{2n+p}/T^{n+p}$ is equal to $n+p$, then N is equivariantly diffeomorphic to the moment angle manifold $\mathcal{Z}_P$.

Theorem 6.1. *Let N^{2n+p} be a simply connected principal T^p-bundle over M^{2n} where M^{2n} has a locally standard, smooth T^n action with orbit space P^n, where P^n is an acyclic manifold with acyclic faces, with $n+p$ facets. We further assume that the bundle of principal orbits is trivial. Then N^{2n+p} is equivariantly diffeomorphic to the moment angle manifold $\mathcal{Z}_P$.*

Proof. The proof is based on work by Davis [7]. Let N^{2n+p} be a principal T^p-bundle over M^{2n}. Such bundles are classified by homotopy classes of maps of M^{2n} into BT^p, denoted by $[M^{2n}, BT^p]$. But

$$[M^{2n}, BT^p] \cong [M, BS^1] \times ... \times [M, BS^1] \cong \oplus_{i=1}^{p} H^2(M^{2n}; \mathbb{Z}).$$

By the short exact sequence in Display (2.1) we know that $H^2(M_T) \longrightarrow H^2(M)$ is onto, hence so is $[M_T, BT^p] \longrightarrow [M, BT^p]$. Thus any principal T^p-bundle over M^{2n} is the pull-back of a principal T^p-bundle over M_T. Note that this statement also holds for subgroups H of T^p. Using a result of Hattori and Yoshida [22], this implies that the T^n-action on M^{2n} lifts to a $T^n \times T^p$-action on N^{2n+p}. Now let $L_i \subset T^n \times T^p = T^{n+p}$ be the isotropy subgroup at a point in the interior of a facet F_i.

To get a T^{n+p} action on N that is modeled on the standard representation we will need the homomorphism $T^{n+p} \longrightarrow T^n \times T^p = T^{n+p}$ that takes the coordinate circles T_i to L_i to be an isomorphism. In general if we have given a collection $L_1, \cdots, L_{n+p}$ of circle subgroups of $T \times H = T^n \times T^p = T^{n+p}$, we say that the L_i span $T \times H$ if the homomorphism $T^{n+p} \longrightarrow T^n \times H$ that takes T_i to L_i is an isomorphism.

We now use Lemma 6.5 of [7] which says that the L_i span $T^n \times H$ if and only if $H_1(N) = 0$. Since N is simply connected by assumption, we obtain that the L_i span $T^n \times T^p$ which implies that N is modeled on the standard representation. By hypothesis the bundle of principal orbits is trivial. We can now apply Proposition 6.2 of [7] to conclude that N is equivariantly diffeomorphic to the moment angle manifold $\mathcal{Z}_P$. The following diagram illustrates this case.

$$\begin{tikzcd}
N^{2n+p} \arrow[r, "\cong"] \arrow[d, "/T^p"] \arrow[dd, bend right, "/T^{n+p}"'] & \mathcal{Z}_P^{2n+p} \arrow[dl, "/T^p"] \\
M^{2n} \arrow[d, "/T^n"] & \\
P^n &
\end{tikzcd}$$

□

Lemma 2.24 tells us that a closed, simply-connected, non-negatively curved torus manifold is locally standard and that M/T and all its faces (after smoothing the corners) are diffeomorphic to disks. In particular, this last fact implies that the bundle of principal orbits is trivial. Thus we may apply Theorem 6.1 to the case at hand. We may then reformulate Theorem 6.1 for the special case where the principal torus bundle is non-negatively curved.

Corollary 6.2. *Let N^{2n+p} be a closed, simply-connected, non-negatively curved principal T^p-bundle over M^{2n}, where M^{2n} is a torus manifold with orbit space P^n as in Display (2.2), with $n+p$ facets. Then N^{2n+p} is equivariantly diffeomorphic to the moment angle manifold $\mathcal{Z}_P$, which is itself a product of spheres.*

6.1.2. **Case (1b): The number of facets is strictly greater than the rank of the torus action.** Here we consider the case where the number of facets of $P^n = M/T$ is strictly greater than the rank of the torus action. In Theorem 6.3 we will show that with this hypothesis N is equivariantly diffeomorphic to the quotient by a free, linear torus action of $\mathcal{Z}_P$.

Theorem 6.3. *Let N^{2n+p} be a simply connected principal T^p-bundle over M^{2n} where M^{2n} has a locally standard, smooth T^n action with orbit space P^n, where P^n is an acyclic manifold with acyclic faces, with m facets and with $m > n+p$. We further assume that the bundle of principal orbits is trivial. Then there is a smooth principal T^{m-n-p}- bundle $\pi : Y^{n+m} \longrightarrow N^{2n+p}$ and if Y^{n+m} is simply connected, it is T^m- equivariantly diffeomorphic to the moment angle manifold Z_P. Then N^{2n+p} is diffeomorphic to the quotient of $\mathcal{Z}_P$ by a T^{m-n-p}-action.*

Proof. Let Y be a principal T^{m-n-p}-bundle over N. Such bundles are classified by homotopy classes of maps of N into BT^{m-n-p}, hence by $[N, BT^{m-n-p}]$. But

$$[N, BT^{m-n-p}] \cong [N, BS^1] \times ... \times [N, BS^1] \cong \oplus_{i=1}^{m-n-p} H^2(N;\mathbb{Z}).$$

Since N is a principal T^p-bundle over M^{2n}, and $H^2(M^{2n}) \cong \mathbb{Z}^{m-n}$, (see for example Theorem 7.4.35 in [4]), using the Leray-Serre spectral sequence or the homotopy sequence for the bundle we obtain that $H^2(N;\mathbb{Z})$ contains $\mathbb{Z}^{m-n-p}$ as a subgroup. We now use the fact that Y is a T^{m-n}-principal bundle over M^{2n} since in the category of finite dimensional manifolds the composition of two principal bundles is again a principal bundle (see, for example, McKay [30]). This means that we are now in the setting of Theorem 6.1 if Y is simply connected. Hence Y^{m+n} is equivariantly diffeomorphic to the moment angle manifold $\mathcal{Z}_P$ as claimed. The following diagram illustrates this case.

$$\begin{tikzcd}
Y^{n+m} \arrow[r, "\cong"] \arrow[d, "/T^{m-n-p}"] & \mathcal{Z}_P^{n+m} \arrow[dl, "/T^{m-n-p}"] \arrow[ddl, bend left, "/T^{m-n}"] \\
N^{2n+p} \arrow[d, "/T^p"] \arrow[dd, bend right, "/T^{n+p}"'] & \\
M^{2n} \arrow[d, "/T^n"] & \\
P^n &
\end{tikzcd}$$

□

As in Case (1a), we now apply Theorem 6.3 to the case of a closed, simply-connected, principal T^p-bundle admitting a family of metrics of almost non-negative curvature over M^{2n}, where M^{2n} is a torus manifold with non-negatively curved orbit space P^n and this principal bundle satisfies Condition F. We obtain the following corollary of Theorem 6.3 by applying Theorems 4.4 and 4.5, noting that the T^{m-n-p} action on the moment angle manifold is a sub-action of the free, linear torus action whose quotient is M^{2n}.

Corollary 6.4. *Let N^{2n+p} be a closed, simply-connected, principal T^p-bundle admitting a family of metrics of almost non-negative curvature over M^{2n}, where M^{2n} is a torus manifold with non-negatively curved orbit space P^n as in Display (2.2), with m facets with $m > n+p$ and N^{2n+p} satisfies Condition F. Then there is a smooth principal T^{m-n-p}-bundle $\pi : Y^{n+m} \longrightarrow N^{2n+p}$. Further, if Y^{n+m} is simply connected, then it is T^m-equivariantly diffeomorphic to the moment angle manifold $\mathcal{Z}_P$, which is itself a product of spheres. Then N^{2n+p} is diffeomorphic to the quotient of $\mathcal{Z}_P$ by a free, linear T^{m-n-p}-action.*

We note that we may state Corollary 6.2 for the class of manifolds considered in Corollary 6.4, as well, but do not, as we will not need it here. We are now ready to prove Case (1) of Theorem A.

Proof of Case (1) of Theorem A. We must show that given an isometric, effective, maximal rank k torus action, M^n is equivariantly diffeomorphic to the quotient by a free, linear torus action of a product of spheres of dimension greater than or equal to three, with the additional hypothesis that the free dimension is equal to the free rank.

By Corollary 5.5 it follows that $M^{2n-2k} = M^n/T^{2k-n}$ admitting a T^{n-k} action is a closed, simply-connected, non-negatively curved torus manifold. Now if $P = M^{2n-2k}/T^{n-k}$ has k facets, then we may apply Corollary 6.2. If it has strictly greater than k facets, we may apply Corollary 6.4 to obtain the result. Hence Case (1) is proven.

We note that for Case (1b), we actually do not need the full power of Corollary 6.4, since we assume that N^{2n+p} is non-negatively curved.

□

6.2. **The proof of Case (2) of Theorem A.** In this section we will prove Case (2) of Theorem A, namely when the free dimension does not equal the free rank. Recall that by Corollary 5.5, $M^n/T^{2k-n} = X^{2n-2k}$ is a non-negatively curved torus orbifold admitting a T^{2n-2k} isometric action.

Applying Theorem 2.25, we see that M^n is rationally elliptic. For quotients of rationally elliptic manifolds admitting almost free torus actions, we have the following observation.

Observation 6.5. *Let M be a closed, rationally elliptic manifold admitting an effective, isometric almost free torus action. Then M/T is a closed, rationally elliptic orbifold.*

The simple-connectivity of M/T follows directly from [2]. The torus action on M induces a fibration, $T \to M \to M/T$, and the long exact sequence in homotopy tells us that $\pi_i(M) \cong \pi_i(M/T)$ for all $i \geq 3$ and then we have

$$0 \to \pi_2(M) \to \pi_2(M/T) \to \pi_1(T) \to 0.$$

Since the quotient space is closed, the condition on the cohomology groups is automatically satisfied.

By Observation 6.5, $M^n/T^{2k-n} = X^{2n-2k}$ is a simply-connected, non-negatively curved, rationally elliptic torus orbifold. Using the proof of Theorem 2.20, it follows that $P = M/T$ is of the form as in Display (2.2).

6.2.1. **Case (2a): The number of facets of M/T is equal to the rank of the torus action.** By Lemma 3.9, we see that all singular isotropies are connected, all singular orbits correspond to boundary points on M/T and all other orbits are principal. It follows by the Equivariant Cross Sectioning Theorem 3.1 that a cross section for the action on M exists. Further, by Proposition 3.11, it follows that M is equivariantly diffeomorphic to $\bar{M}$, a closed, simply connected non-negatively curved (with the pullback metric) Riemannian manifold admitting an isometric, effective action of T^k with free dimension $2k - n$. This in turn tells us that $\bar{M}^{2n-2k} = \bar{M}/T^{2k-n}$ is a torus manifold of non-negative curvature and such that $P = M/T$ is of the form as in Display (2.2).

The following diagram illustrates this case.

$$\begin{tikzcd}
M^n \arrow[r, "\cong"] \arrow[d, "/T^{2k-n}"] \arrow[dd, bend right, "/T^k"'] & \bar{M}^n \arrow[r, "\cong"] \arrow[d, "/T^{2k-n}"] & \mathcal{Z}_{\bar{P}^{n-k}} \arrow[dl, "/T^{2k-n}"] \\
X^{2n-2k} \arrow[d, "/T^{n-k}"] & \bar{M}^{2n-2k} \arrow[d, "/T^{n-k}"] & \\
P^{n-k} \arrow[r, "\cong"] & \bar{P}^{n-k} &
\end{tikzcd}$$

6.2.2. **Case (2b): The number of facets of M/T is strictly greater than the rank of the torus action.** Let $k+l=r$, with $l>0$ be the number of facets of P. It follows from the proof of Theorem 2.20, that M has the rational homotopy type of the base of a principal T^l bundle with total space the corresponding moment angle manifold, which is a product of spheres of dimensions greater than or equal to three. In particular, using the long exact sequence in homotopy, this tells us that $H^2(M^n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^l$. Note that principal T^l-bundles over M are classified by l elements $\beta_1,\ldots,\beta_l \in H^2(M;\mathbb{Z})$. Here each β_i can be described as the Euler class of the oriented circle bundle $N/T^{l-1} \longrightarrow M$ where N is the total space and $T^{l-1} \subset T^l$ is the subtorus with the i-th S^1-factor deleted. It follows that there exists a T^l-principal bundle over M^n with total space N^{n+l}, a closed, simply-connected Riemannian manifold. Further by Theorem 2.28, N^{n+l} admits a family of almost non-negatively curved metrics and by construction, its quotient space is non-negatively curved and N^{n+l} satisfies Condition F.

As in Case (2a), using Lemma 3.9, the Equivariant Cross Sectioning Theorem 3.1 and Proposition 3.11, it follows that N^{n+l} is equivariantly diffeomorphic to $\bar{N}^{n+l}$ and $\bar{N}^{n+l}$ admits an isometric and effective T^{k+l}-action with free dimension equal to $2k+l-n$. Moreover, $\bar{N}^{n+l}$ (with the pullback metric) admits a family of almost non-negatively curved metrics, its quotient space is non-negatively curved and $\bar{N}^{n+l}$ satisfies Condition F. Hence $\bar{M}^{2n-2k} = \bar{N}^{n+l}/T^{2k+l-n}$ is a torus manifold admitting a family of almost non-negatively curved metrics, with non-negatively curved quotient space, satisfying Condition F.

The following diagram illustrates this case.

$$\begin{tikzcd}
N^{n+l} \arrow[r, "\cong"] \arrow[d, "/T^{l}"] & \bar{N}^{n+l} \arrow[r, "\cong"] \arrow[dd, "/T^{2k-n+l}"] & \mathcal{Z}_{\bar{P}^{n-k}} \arrow[ddl, "/T^{2k-n+l}"] \\
M^n \arrow[d, "/T^{2k-n}"] \arrow[dd, bend right, "/T^k"'] & & \\
X^{2n-2k} \arrow[d, "/T^{n-k}"] & \bar{M}^{2n-2k} \arrow[d, "/T^{n-k}"] & \\
P^{n-k} \arrow[r, "\cong"] & \bar{P}^{n-k} &
\end{tikzcd}$$

We are now ready to prove Case (2) of Theorem A.

Proof of Case (2) of Theorem A. We must show that given an isometric, effective, maximal rank k torus action, M^n is equivariantly diffeomorphic to the quotient by a free, linear torus action of a product of spheres of dimension greater than or equal to three.

In case (2a), $\bar{M}^{2n-2k}$ is a torus manifold of non-negative curvature, and in case (2b), $\bar{M}^{2n-2k}$ is a torus manifold admitting a family of almost non-negatively curved metrics, with non-negatively curved quotient space, and satisfying Condition F.

Thus, in case (2a) we may apply Corollary 6.2 to show that $\bar{M}$ is equivariantly diffeomorphic to the moment angle manifold $\mathcal{Z}_P$ as in Display (2.3) and hence M itself is equivariantly diffeomorphic to $\mathcal{Z}_P$ as desired.

Likewise, in case (2b), we can apply Corollary 6.4 to conclude that $\bar{M}^{2n-2k}$ is the quotient of a free torus action on $\bar{N}^{n+l}$, which in turn is equivariantly diffeomorphic to the moment angle manifold $\mathcal{Z}_P$. Since $\bar{N}^{n+l}$ is equivariantly diffeomorphic to N^{n+l}, by commutativity of the diagram, we can then conclude that M is equivariantly diffeomorphic to the quotient of a free torus action on the moment angle manifold $\mathcal{Z}_P$, a product of spheres of dimensions greater than or equal to three.

It remains to show that the T^l action is linear. The following diagram illustrates the proof. Use the cross-sections c_1 and c_2 to construct a diffeomorphism from M^n to $\bar{M}^n$. Then $M^n \cong \bar{M}^n \cong \mathcal{Z}_{\tilde{P}^{n-k}}/T^l$ and T^l is a free linear action since it is sub-action of the free linear action of T^{2k-n+l} on $\mathcal{Z}_{\tilde{P}^{n-k}}$.

$$\begin{tikzcd}
N^{n+l} \arrow[r, "\cong"] \arrow[d, "/T^l"] & \bar{N}^{n+l} \arrow[r, "\cong"] \arrow[d, "/T^l"] \arrow[dd, bend left] & \mathcal{Z}_{\bar{P}^{n-k}} \arrow[ddl, "/T^{2k-n+l}"] \\
M^n \arrow[r, dashed, "\cong"] \arrow[d, "/T^{2k-n}"] & \bar{M}^n \arrow[d] & \\
X^{2n-2k} \arrow[d, "/T^{n-k}"] & \bar{M}^{2n-2k} \arrow[d, "/T^{n-k}"] & \\
P^{n-k} \arrow[r, "\cong"] \arrow[uuu, bend left, "c_1"] & \bar{P}^{n-k} \arrow[uuu, bend left, "c_2"] &
\end{tikzcd}$$

□

7. The Proofs of the remaining results

We now present proofs of Corollary B and Theorem D, as well as a streamlined proof of the Maximal Symmetry Rank Conjecture in dimensions less than or equal to 6. We begin with a proof of Corollary B.

Proof of Corollary B. Note that while we can simply appeal to results in [13] to prove the upper bound on the rank, we will give a constructive proof, as it is straightforward and quite simple.

We assume then that $k = \lfloor 2n/3 \rfloor + s$, with $s > 0$ to obtain a contradiction. Since the action is maximal, the free rank is equal to $2k - n$ and in particular, we see that $X^{2n-2k} = M^n/T^{2k-n}$ is a torus orbifold. By Theorem A, M^n is equivariantly diffeomorphic to the free linear quotient by a torus of a product of spheres of dimension greater than three. This product of spheres can have dimension at most $3n - 3k$. In particular, n must be less than or equal to $3n - 3k$. However, a simple calculation shows that for $s \geq 1$,

$$3n - 3k \leq n + 2 - 3s < n,$$

which gives us a contradiction. Hence $k \leq \lfloor 2n/3 \rfloor$, as desired. □

We now note that the proof of Theorem D follows immediately by combining Theorem C with the following proposition.

Proposition 7.1. *Let M^n be a closed, simply-connected, non-negatively curved Riemannian manifold admitting an isometric, effective cohomogeneity three torus action. If $n \geq 7$, then the action is maximal.*

Proof. We first recall the following lemma and corollary from [16].

Lemma 7.2. [16] *Let T^n act on M^{n+3}, a closed, simply-connected smooth manifold. Then some circle subgroup has non-trivial fixed point set.*

Corollary 7.3. [16] *Let M^{n+3} be a closed, non-negatively curved manifold with an isometric T^n action. Suppose that $M^* = S^3$ and that there are isolated T^{n-1} orbits. Then there are at most four such isolated T^{n-1} orbits. In particular, if $n \geq 4$, then there are none.*

Lemma 7.2 tells us that a cohomogeneity three torus action on a closed, simply-connected manifold must have isotropy subgroups of rank at least 1. An effective cohomogeneity k torus action can have isotropy subgroups of rank at most k, and an action with isotropy T^k or T^{k-1} will be maximal or almost maximal, respectively. In order to prove that a cohomogeneity three action must be maximal or almost maximal, we claim that it suffices to show that there can be no action with only isolated circle isotropy.

In fact, if there is only circle isotropy and it is not isolated, then it follows that there is a circle acting fixed point homogeneously. The corresponding codimension two fixed point set of the circle, N^{n+1}, admits an induced, $T^n/T^1 = T^{n-1}$ action of cohomogeneity two. This action is either free or almost free. By Theorems 12.3 and 12.15 of Conner and Raymond [6] (see also [44]), for the action (T^{n-1}, N^{n+1}), with $n \geq 4$, one can find a splitting $T^{n-1} = T^{n-2} \times T^1$ and a finite abelian subgroup $\Gamma \subset T^{n-2}$ so that (T^{n-2}, N^{n+1}) is fibered equivariantly over $(T^{n-2}, T^{n-2}/\Gamma)$, with fiber M^3 and structure group Γ. Further, since N^{n+1} is non-negatively curved, it follows that $\pi_1(M^3)$ is finite (see [42]). Using the low degree terms in the Leray-Serre-Atiyah-Hirzebruch sequence, see [9], we obtain that $\pi_* : H_1(N^{n+1}) \longrightarrow H_1(T^{n-2})$ is surjective, where $\pi : N^{n+1} \longrightarrow T^{n-2}$ is the projection map. Hence $\mathrm{rk}(H_1(N^{n+1})) \geq n-2 > 1$, a contradiction to Lemma 5.2.

Recall from Corollary 4.7 of Chapter IV of [2] that the quotient space, M^*, of a cohomogeneity three G-action on a compact, simply-connected manifold with connected orbits is a simply-connected 3-manifold with or without boundary. Note that when there is only isolated circle isotropy for a cohomogeneity three torus action, the quotient space will not have boundary and thus, by the resolution of the Poincaré conjecture (see Perelman [37, 38, 39]), we have that $M^* = S^3$. Therefore, we may apply Corollary 7.3 and Proposition 7.1 follows. □

Finally, we present a significantly streamlined proof of the Maximal Symmetry Rank Conjecture for dimensions less than or equal to 6 (cf. [15], [12]), that is we will prove the following Theorem.

Theorem 7.4. *Let M^n, $2 \leq n \leq 6$ be a closed, simply-connected, non-negatively curved manifold admitting an effective, isometric torus action. Then the Maximal Symmetry Rank Conjecture holds.*

Proof. We begin with the following observation.

Observation 7.5. *For a cohomogeneity one torus action on a closed, simply-connected manifold of dimension $n \geq 2$, without curvature restrictions, the action is maximal. This result also holds for a cohomogeneity two torus action on a closed, simply-connected n-manifold, for $n \geq 4$ (see Theorem 1.3 in [24]).*

Combining Theorem C with Observation 7.5, we obtain Theorem 7.4 (cf. [15], [12]). □

One notes that in order to extend these classification results to higher dimensions, it suffices to show that an action of maximal symmetry rank must be either almost maximal or maximal. One possible course of action would be to establish the existence of an upper bound on the free rank of the action that is strictly less than the rank of the action; that is, show that some circle has non-empty fixed point set. However, to date, there are no known obstructions for a cohomogeneity m torus action, $m \geq 4$, on an n-dimensional closed, simply-connected, non-negatively curved Riemannian manifold with $n \geq 10$ to be free (see Kobayashi and Nomizu [26], Angulo-Ardoy, Guijarro and Walschap [1]).

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