A DIFFEOMORPHISM CLASSIFICATION OF GENERALIZED WITTEN MANIFOLDS

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Abstract. We give a classification of a specific family of seven dimensional manifolds, the generalized Witten manifolds up to homeomorphism and diffeomorphism. Using an approach suggested by J. Shaneson, we develop a modified surgery theory which fully classifies these manifolds. In contrast to previous approaches, this surgery theory is designed to be more readily applicable to higher dimensions. The family contains examples of manifolds which admit Einstein metrics and are homeomorphic but not diffeomorphic. These particular manifolds occur naturally in differential geometry and are of great interest to both differential geometers and physicists.

1. Introduction

About thirty years ago differential geometers began asking questions about the topological nature of a fundamental class of manifolds, namely homogeneous spaces. Homogeneous spaces are Riemannian manifolds which admit a transitive Lie group action. They comprise a rich set of examples in Riemannian geometry and theoretical physics.

Of particular interest is a diffeomorphism classification of homogeneous Einstein manifolds. Constructing diffeomorphic Einstein manifolds of volume one with different Einstein constants leads to information about the number of components in the moduli space of Einstein metrics.

In developing Kaluza-Klein theories [DNP] theoretical physicists also studied specific families of homogeneous Einstein manifolds. In particular, E. Witten [W] studied seven dimensional homogeneous spaces with symmetry group $SU(3) \times SU(2) \times U(1)$. In fact, these spaces are S^1-bundles over $\mathbb{C}P^2 \times \mathbb{C}P^1$. We will call them Witten manifolds. In 1988, Kreck and Stolz [KS1] stated a diffeomorphism and homeomorphism classification for this family of Witten manifolds and used it to find examples of Witten manifolds which are homeomorphic but not diffeomorphic. The necessary machinery for the classification was developed by M. Kreck and published in 1999, [Kr1]. During a seminar in 1990, J. Shaneson [Sh] outlined a different approach to this type of classification. We use this basic

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outline to independently classify a more general class of manifolds that includes the Witten manifolds. Recently, our approach was briefly discussed in a survey article on these types of classifications by M. Kreck [Kr2].

The Witten manifolds, $M_{k,l}$, are orbit spaces of S^1- actions on $S^5 \times S^3$ parametrized by two non-zero coprime integers (k,l). Using the standard circle actions on S^5 and S^3, the S^1- action can be described as follows. For $X \in S^5 \subset \mathbb{C}^3$ and $Y \in S^3 \subset \mathbb{C}^2$, $z \in S^1$ acts by $(X,Y) \mapsto (z^k X, z^l Y)$. There is a natural generalization of this family of Witten manifolds, first used in Kruggel [K1], by considering orbit spaces of the following S^1- action on $S^5 \times S^3$.

$$\begin{aligned} &S^1 \times S^5 \times S^3 \longrightarrow S^5 \times S^3 \\ &(z,(u_1,u_2,u_3),(v_1,v_2)) \longmapsto (z^k u_1, z^k u_2, z^k u_3),(z^{l_1} v_1, z^{l_2} v_2)), \end{aligned} \tag{1}$$

where $S^5 \subset \mathbb{C}^3$, $S^3 \subset \mathbb{C}^2$ and k and l_j are non-zero integers for $j \in \{1,2\}$. This action is free if and only if $\gcd(k\,,\,l_j) = 1$ for $j \in \{1,2\}$. Hence we obtain the *generalized Witten manifolds* $N_{k,l} := (S^5 \times S^3)/S^1$ where $l = (l_1\,,l_2)$ and $\gcd(k\,,\,l_j) = 1$ for $j \in \{1,2\}$.

Our main result is a homeomorphism and diffeomorphism classification of the generalized Witten manifolds. Although our Theorem 1 can be deduced from [KS2, Theorem 3.1] — which is mainly based on [KS2, Proposition 3.2] which in turn was proven in [Kr1, Theorem G]—, our proof is independent of both [KS2] and [Kr1]. It seems likely that our methods can be used to obtain their result.

Although we do not use this here, we note that the homotopy equivalence classification for the generalized Witten manifolds has been given by [K1].

Even though the Witten manifolds are Einstein manifolds it is not known whether there exist Einstein metrics on the generalized Witten manifolds.

We use a set of six invariants $s_i(N_{k,l})$ and $\bar{s}_i(N_{k,l})$ which are described in Section 4. They are based on the Eells-Kuiper invariant [EK] and can be expressed as linear combinations of eta-invariants. For $i = 1$ one obtains exactly the Eells-Kuiper μ-invariant in the spin case, whereas the invariants s_2 and s_3 are modifications of s_1 and were introduced in [KS1].

Theorem 1. *Let $N_{k,l}$ and $N_{k',l'}$ be two generalized Witten manifolds which are both spin or both non-spin. If $l_1 \neq l_2$, assume further that for each of the primes $p = 2,3$ either gcd$(p,l_i) = 1$ for $i = 1,2$ or gcd$(p,l_i) \neq 1$ for $i = 1,2$. Then $N_{k,l}$ is diffeomorphic (homeomorphic) to $N_{k',l'}$ if and only if*

$$\begin{aligned} &|H^4(N_{k,l};\mathbb{Z})| = |H^4(N_{k',l'};\mathbb{Z})| \textit{ and} \\ &s_i(N_{k,l}) = s_i(N_{k',l'}) \in \mathbb{Q}/\mathbb{Z} \\ &(\textit{resp. } \bar{s}_i(N_{k,l}) = \bar{s}_i(N_{k',l'}) \in \mathbb{Q}/\mathbb{Z}\,) \textit{ for } i = 1,2,3\,. \end{aligned}$$

Remark 1. The case of p dividing exactly one of the l_i requires a substantially different cobordism calculation and will be treated in a sequel to this paper.

In order to prove Theorem 1 we develop a general technique for the classification of families of manifolds up to homeomorphism and diffeomorphism. This technique can be applied both to manifolds of the same cohomology ring as the generalized Witten manifolds and to more general manifolds. In particular, it can be applied to higher dimensional homogeneous spaces that are of interest in differential geometry and theoretical physics. The technique consists of three basic steps. **Step 1** : we compute the full cobordism set of normal maps into a topological model space $E_{l_1 l_2}$. The construction of the space $E_{l_1 l_2}$ is described in Section 2. In general this model space depends on the cohomology of the manifolds to be classified. **Step 2** : we show that the structure set of the surgery exact sequence embeds into the set of normal maps. For this step we must use a modified surgery theory, due to the fact that our specific model space is not a Poincaré duality space. **Step 3** : we show that the group generated by the invariants is isomorphic to the set of normal maps. Combining steps two and three proves the uniqueness and existence of the surgery problem and completes the classification theorem. In general, it is necessary to find the correct model space and suitable invariants in order to use this technique.

The following theorem summarizes Step 1 for both the differentiable and topological category.

Theorem 2. *If* $l_1 = 2^{m_1} 3^{m_2} l_1'$ *and* $l_2 = 2^{n_1} 3^{n_2} l_2'$ *where* m_1, m_2, n_1, n_2 *are nonnegative integers and* $(6, l_1' l_2') = 1$, *then we obtain for the full set of normal cobordism classes of normal maps in the differentiable category*

$$NM_O(E_{l_1 l_2}, \xi_k) \otimes \mathbb{Z}_2 = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_4 \ \textit{if}\ m_1 = 0\,,\ n_1 = 0\,,\ k \ \textit{odd} \\ \mathbb{Z}_4 \oplus \mathbb{Z}_4 \ \textit{if}\ m_1 = 0\ ,n_1 = 0\,,\ k \ \textit{even} \\ \mathbb{Z}_4 \oplus \mathbb{Z}_4 \ \textit{if}\ m_1 > 0\,,\ n_1 > 0 \end{cases}$$

$$NM_O(E_{l_1 l_2}, \xi_k) \otimes \mathbb{Z}_3 = \begin{cases} \mathbb{Z}_3 \ \textit{if}\ m_2 > 0\,,\ n_2 > 0 \\ 0 \quad \textit{if}\ m_2 = 0\,,\ n_2 = 0 \end{cases}$$

$$NM_O(E_{l_1 l_2}, \xi_k) \otimes \mathbb{Z}_7 = \mathbb{Z}_7.$$

and in the topological category

$$NM_{TOP}(E_{l_1 l_2}, \xi_k) \otimes \mathbb{Z}_2 = \begin{cases} \mathbb{Z}_2 \ \textit{if}\ m_1 = 0\,,\ n_1 = 0\,,\ k \ \textit{odd} \\ \mathbb{Z}_4 \ \textit{if}\ m_1 = 0\ ,n_1 = 0\,,\ k \ \textit{even} \\ \mathbb{Z}_4 \ \textit{if}\ m_1 > 0\,,\ n_1 > 0 \end{cases}$$

$$NM_{TOP}(E_{l_1 l_2}, \xi_k) \otimes \mathbb{Z}_3 = \begin{cases} \mathbb{Z}_3 \ \textit{if}\ m_2 > 0\,,\ n_2 > 0 \\ 0 \quad \textit{if}\ m_2 = 0\,,\ n_2 = 0\,. \end{cases}$$

Here ξ_k is the normal bundle of the model space $E_{l_1 l_2}$. Our model space $E_{l_1 l_2}$ is not a Poincaré space and hence we cannot apply traditional surgery theory. Our idea is to generalize surgery to this setting by requiring a homotopy equivalence between $E_{l_1 l_2}$ and $N_{k,l}$ only up to the middle dimension. As the fourth homology group of the generalized Witten manifolds $N_{k,l}$ vanishes, it is possible to generalize surgery to this case.

Theorem 3. *The structure set of the model space $E_{l_1 l_2}$ embeds into the set of normal cobordism classes of normal maps, i.e. $\mathcal{S}_{TOP}(E_{l_1 l_2}) \hookrightarrow NM_{TOP}(E_{l_1 l_2})$ for the topological category and $\mathcal{S}_O(E_{l_1 l_2})/\theta_7 \hookrightarrow NM_O(E_{l_1 l_2})$ for the differentiable category where θ_7 is the group of exotic 7-spheres.*

Using the invariants $s_i(N_{k,l})$, $i = 1,2,3$, for the differentiable category and the invariants $\bar{s}_i(N_{k,l})$, $i = 1,2,3$, for the topological category, we obtain the following theorem.

Theorem 4. *The group generated by the invariants $s_i(N_{k,l}) - s_i(N_{k',l'})$ for $i = 1,2,3$ (resp. $\bar{s}_i(N_{k,l}) - \bar{s}_i(N_{k',l'})$ for $i = 1,2,3$) is isomorphic to the group of normal cobordism classes of normal maps $NM_O(E_{l_1 l_2})$ in the differentiable category (resp. $NM_{TOP}(E_{l_1 l_2})$ in the topological category).*

Note that Theorem 4 implies that there are sufficiently many examples of manifolds $N_{k,l}$ to generate the full cobordism ring. Together with Theorems 2 and 3 we obtain existence and uniqueness of the classification problem and Theorem 1 follows.

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2. Geometric and topological preliminaries

2.1. Description of Witten manifolds $\mathbf{M_{k,l}}$ and their generalization $\mathbf{N_{k,l}}$.

We will use a description of the spaces $M_{k,l}$ given by Witten. The manifolds are orbit spaces of S^1-actions on $S^5 \times S^3$ parametrized by two integers (k,l) where $z \in S^1$ acts by $(X,Y) \mapsto (z^k X, z^l Y)$ for $X \in S^5$ and $Y \in S^3$. Here k and l are assumed to be non-zero, for otherwise we obtain a different type of manifold. Moreover, as we are interested in the simply connected case, the integers k and l must be coprime. (For all this information, see [W]).

These manifolds $M_{k,l}$ can equivalently be described as the total space of an S^1-bundle over $\mathbb{C}P^2 \times \mathbb{C}P^1$ with first Chern class equal to $-l\,x + k\,y$ where x is a generator of $H^2(\mathbb{C}P^2)$ and y of $H^2(\mathbb{C}P^1)$. For different k and l, non-zero and relatively prime, one obtains all seven dimensional simply connected homogeneous manifolds with $SU(3) \times SU(2) \times U(1)$ symmetry [DNP].

Our goal is to classify the following generalization of this family of Witten manifolds. The manifolds $N_{k,l}$ are still orbit spaces of S^1-actions on $S^5 \times S^3$, however they are now parametrized by three integers (k, l_1, l_2) where $z \in S^1$ acts by $(X,Y) \mapsto (z^k X, z^{l_1} Y_1, z^{l_2} Y_2)$ for $X \in S^5$ and $Y = (Y_1, Y_2) \in S^3$. Again k, l_1, l_2 are assumed to be non-zero and we denote $l = (l_1, l_2)$ to simplify the notation. Again, for the simply connected case, the integers k and l_1 and k and l_2 must be coprime. A similar generalization using different integers for the action on S^5 as well as for the action on S^3 was used for a homotopy equivalence classification of this family of manifolds in [K1].

Due to the generalization the manifolds $N_{k,l}$ can no longer be described as the total space of an S^1-bundle over $\mathbb{C}P^2 \times \mathbb{C}P^1$. However, we still obtain a bundle construction over $\mathbb{C}P^2$: the $N_{k,l}$ are the total spaces of fiber bundles with fiber the lens spaces $L_k(l_1, l_2)$ and structure group S^1.

2.2. **Cohomology ring of generalized Witten manifolds $\mathbf{N_{k,l}}$.** We use the definition of $N_{k,l}$ as the orbit space of the S^1-action described above. Note that we can write $S^5 \times S^3$ as the following quotient space.

$$S^5 \times S^3 = [U(3) \times U(2)] \,/\, [U(2) \times U(1)]\,.$$

If we embed S^1 into $[U(3) \times U(2)]$ as

$$z \mapsto \left[\begin{pmatrix} z^k & & \\ & z^k & \\ & & z^k \end{pmatrix} , \begin{pmatrix} z^{l_1} & \\ & z^{l_2} \end{pmatrix} \right],$$

then we can view $N_{k,l}$ as a biquotient.

$$N_{k,l} = S^1 \backslash [U(3) \times U(2)] \,/\, [U(2) \times U(1)]\,.$$

Following [S] we obtain the following diagram of fibrations.

$$\begin{array}{ccc} S^5 \times S^3 & & S^5 \times S^3 \\ \downarrow & & \downarrow \\ N_{k,l} & \longrightarrow & B(U(2) \times U(1)) \\ \downarrow & & \downarrow \\ BS^1 & \longrightarrow & B(U(3) \times U(2)) \end{array}$$

Using the Serre spectral sequence for the left fibration, we obtain that the only possible non-trivial differentials are d^4 and d^6. We now employ the Serre spectral sequence for the right fibration and use naturality of the pull-back to compute these differentials. Hence $E^2 = E^3 = E^4$ and $d^4(x_3) = l_1\, l_2\, u^2$ where x_3 is a generator of $H^3(S^3;\mathbb{Z})$ and u^2 is a generator of $H^4(BS^1;\mathbb{Z})$. We again use naturality of the pull-back to obtain $E^\infty = E^7 = H(E^6, d^6)$ with $d^6(x_5) = u^3$ where x_5 is a generator of $H^5(S^5;\mathbb{Z})$. Using these differentials in the Serre spectral sequence one obtains for the cohomology ring of $N_{k,l}$ with integer coefficients:

$$H^0(N_{k,l};\mathbb{Z}) \cong \mathbb{Z}, \qquad H^1(N_{k,l};\mathbb{Z}) \cong 0, H^2(N_{k,l};\mathbb{Z}) \cong \mathbb{Z}, H^3(N_{k,l};\mathbb{Z}) \cong 0,$$
$$H^4(N_{k,l};\mathbb{Z}) \cong \mathbb{Z}_{|l_1 l_2|}, H^5(N_{k,l};\mathbb{Z}) \cong \mathbb{Z}, H^6(N_{k,l};\mathbb{Z}) \cong 0,\ H^7(N_{k,l};\mathbb{Z}) \cong \mathbb{Z}\,.$$

Here $H^4(N_{k,l};\mathbb{Z})$ is generated by the square of the generator of $H^2(N_{k,l};\mathbb{Z})$.

2.3. **Description of model space $\mathbf{E_{l_1 l_2}}$.** Recall that there is an isomorphism between all maps of a topological space X into an Eilenberg-Maclane space, $[X\,,\,K(\mathbb{Z},n)]$, and the n-th cohomology group of X with integer coefficients. In particular, using the fact that

$$[N_{k,l}\,,\,K(\mathbb{Z},2)] \cong H^2(N_{k,l};\mathbb{Z}) \cong \mathbb{Z} \text{ and } [N_{k,l}\,,\,K(\mathbb{Z},4)] \cong H^4(N_{k,l};\mathbb{Z}) \cong \mathbb{Z}_{l_1 l_2}\,,$$

we obtain the following diagram:

$$\begin{tikzcd} & K(\mathbb{Z},3) \arrow[d] & K(\mathbb{Z},3) \arrow[d] \\ & E_{l_1 l_2} \arrow[r] \arrow[d] & P(K(\mathbb{Z},4)) \arrow[d] \\ N_{k,l} \arrow[r, "g"] & K(\mathbb{Z},2) \arrow[r, "f"] & K(\mathbb{Z},4) \end{tikzcd}$$

Here $P(K(\mathbb{Z},4))$ is the path space of $K(\mathbb{Z},4)$ and $E_{l_1 l_2}$ is the total space of the pull back bundle under the map f which is multiplication by $l_1\, l_2$. Using the fact that $H^3(N_{k,l}) \cong 0$ we obtain a unique lift $\phi : N_{k,l} \longrightarrow E_{l_1 l_2}$ of the map g. We will use this map for our surgery set-up. The total space $E_{l_1 l_2}$ will be our model space, i.e. the space we map into via normal maps. The map ϕ is a homotopy equivalence up to dimension four, indeed it induces an isomorphism up to the fourth cohomology.

2.4. **Cohomology ring of model space $\mathbf{E_{l_1 l_2}}$.** Again we use the Serre spectral sequence to compute the cohomology ring of $E_{l_1 l_2}$ as $E_{l_1 l_2}$ is the total space of a $K(\mathbb{Z},3)$ bundle over $K(\mathbb{Z},2)$. To complete this calculation we recall the cohomology ring of the Eilenberg-Maclane spaces $K(\mathbb{Z},2)$ and $K(\mathbb{Z},3)$. As the infinite complex projective space $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$ we obtain that the cohomology ring of $K(\mathbb{Z},2)$ with integer coefficients is a polynomial algebra over $\mathbb{Z}$ with a generator in dimension 2.

$$H^*(K(\mathbb{Z},2);\mathbb{Z}) = H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[x]\ ,\ \dim(x) = 2\,.$$

To compute the cohomology ring of $K(\mathbb{Z},3)$ we use the Serre spectral sequence of the path fibration

$$\begin{tikzcd} K(\mathbb{Z},2) \arrow[r] & P(K(\mathbb{Z},3)) \arrow[d] \\ & K(\mathbb{Z},3) \end{tikzcd}$$

and obtain for the first few cohomology groups of $K(\mathbb{Z},3)$

$$H^q(K(\mathbb{Z},3);\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ generated by } 1 & \text{if } q=0\,, \\ \mathbb{Z} \text{ generated by } s & \text{if } q=3\,, \\ \mathbb{Z}_2 \text{ generated by } s^2 & \text{if } q=6\,, \\ \mathbb{Z}_3 \text{ generated by } t & \text{if } q=8\,, \\ 0 & \text{if } q=1,\ 2,\ 4,\ 5,\ 7\,. \end{cases}$$

Substituting these into the Serre spectral sequence for $E_{l_1l_2}$ and using the universal coefficient theorem to deal with an extension problem in dimensions 6 and 8, we derive the first eight cohomology groups of $E_{l_1l_2}$ with integer coefficients.

$$H^q(E_{l_1l_2};\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q=0,\ 2, \\ \mathbb{Z}_{|l_1l_2|} & \text{if } q=4\,, \\ \mathbb{Z}_{|l_1l_2|}\oplus\mathbb{Z}_2 & \text{if } q=6\,, \\ \mathbb{Z}_{|l_1l_2|}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_3 & \text{if } q=8 \\ 0 & \text{if } q=1,\ 3,\ 5,\ 7\,. \end{cases}$$

2.5. **Stable normal bundle.** Recall that the manifolds $M_{k,l}$ can be viewed as S^1-bundles over $\mathbb{C}P^2\times\mathbb{C}P^1$:

$$\begin{array}{ccc} S^1 & \longrightarrow & M_{k,l} \\ & & \big\downarrow{\scriptstyle \pi} \\ & & \mathbb{C}P^2\times\mathbb{C}P^1 \end{array}$$

By construction, the tangent bundle of the total space $M_{k,l}$ splits into a direct sum of the pull-back of the tangent bundle of the base space and a trivial bundle Σ coming from the tangent bundle of S^1. Hence

$$\tau_{M_{k,l}} \cong \pi^*(\tau_{\mathbb{C}P^2\times\mathbb{C}P^1})\oplus\Sigma$$

As we are looking at stable tangent bundles only, we can compose π with a further projection to $\mathbb{C}P^2$:

$$\begin{array}{ccc} S^1 & \longrightarrow & M_{k,l} \\ & & \big\downarrow{\scriptstyle \omega} \\ & & \mathbb{C}P^2 \end{array}$$

Therefore the stable tangent bundle of $M_{k,l}$ can be described as

$$\tau^S_{M_{k,l}} \cong_s \omega^*(\tau_{\mathbb{C}P^2})\oplus\Sigma \cong_s 3\cdot\gamma^{\otimes k}\oplus\Sigma$$

where γ is the canonical line bundle over $\mathbb{C}P^2$ with first Chern class $c_1(\gamma^{\otimes k}) = k\cdot u$ where u is a generator of $H^2(M_{k,l};\mathbb{Z})$. Also note that in this case $w_2(\tau_{M_{k,l}}) =$

$c_1(\tau_{M_{k,l}}) = e(\tau_{M_{k,l}}) = e(3 \cdot \gamma^{\otimes k} \oplus \Sigma) = k \cdot u \in \mathbb{Z}_2$. This enables us to compute the Pontrjagin and Stiefel-Whitney classes of $M_{k,l}$:

$$\begin{aligned} p_1(\tau_{M_{k,l}}) &= 3\,k^2\,u^2 \;\in \mathbb{Z}_{l^2} \\ w_2(\tau_{M_{k,l}}) &= k\,u \;\in \mathbb{Z}_2\,. \end{aligned}$$

As $\mathbb{C}P^2 \subset \mathbb{C}P^\infty$, we obtain a complex three-plane bundle $3\,\gamma^{\otimes k} \oplus \Sigma$ over $\mathbb{C}P^\infty$ where γ is the tautological bundle. We now pull this bundle back to E_{l^2} and obtain a bundle ξ_k over E_{l^2} . The bundle ξ_k over E_{l^2} is a bundle over the model space E_{l^2} that pulls back to the stable tangent bundle of $M_{k,l}$. In order to use surgery theory for the classification we need the following assumption.

Condition 1. *Assume that $M_{k,l}$ is homotopy equivalent to $M_{k',l'}$ and ξ_k is isomorphic to $\xi_{k'}$ over the 4-skeleton. Then*

$$\begin{aligned} & (i)\ H^4(M_{k,l}) \cong H^4(M_{k',l'}) \\ & (ii)\ \xi_k \cong \xi_{k'}\ \textit{over the } 4-\textit{skeleton} \\ \iff & (i')\ \mathbb{Z}_{l^2} \cong \mathbb{Z}_{l'^2} \\ & (ii')\ p_1(\xi_k) = p_1(\xi_{k'})\,;\ w_2(\xi_k) = w_2(\xi_{k'}) \\ \iff & (i'')\ l = \pm l' \\ & (ii'')\ 3\,k^2 \equiv 3\,k'^2 \mod l^2\,;\ k \equiv k' \mod 2 \end{aligned}$$

As we are interested in the stable normal bundle over E_{l^2} we take the virtual inverse to ξ_k over E_{l^2} and denote it by ξ_k as well.

The generalized Witten manifolds are fiber bundles

$$\begin{array}{ccc} L_k(l_1, l_2) & \longrightarrow & N_{k,l} \\ & & \big\downarrow \omega \\ & & \mathbb{C}P^2 \end{array}$$

with structure group S^1. Now we can also describe $N_{k,l}$ as $[S^5 \times L_k(l_1, l_2)]/S^1$ where $z \in S^1$ acts on $S^5 \times L_k(l_1, l_2)$ as

$$(X, [Y_1, Y_2]) \mapsto (z\,X, [z^{\frac{l_1}{k}}\,Y_1, z^{\frac{l_2}{k}}\,Y_2])$$

where $[Y_1, Y_2] \in L_k(l_1, l_2)$. Here $N_{k,l}$ has been expressed as the associated fiber bundle to the standard Hopf bundle $S^1 \to S^5 \to \mathbb{C}P^2$ with fiber $L_k(l_1, l_2)$. The S^1-action on the fiber $L_k(l_1, l_2)$ is given by $z\,[Y_1, Y_2] = [z^{\frac{l_1}{k}}\,Y_1, z^{\frac{l_2}{k}}\,Y_2]$. Hence we obtain the bundle $S^1 \to S^5 \times L_k(l_1, l_2) \to N_{k,l}$. Note however that the action of S^1 on $L_k(l_1, l_2)$ is in general not free.

Proposition 1. *The fixed point set of the S^1-action on $L_k(l_1, l_2)$ consists of two cyclic groups lying in the circles $[0, Y_2]$ and $[Y_1, 0]$ in $L_k(l_1, l_2)$:*

$$Fix(S^1, L_k(l_1, l_2)) = \begin{cases} \mathbb{Z}_{l_1} \subset S^1/\mathbb{Z}_p \; in \; [Y_1, 0] \subset L_k(l_1, l_2) \\ \mathbb{Z}_{l_2} \subset S^1/\mathbb{Z}_p \; in \; [0, Y_2] \subset L_k(l_1, l_2) \end{cases}$$

where $(l_1, l_2) = p$.

Proof. First observe that

$$e^{\imath\theta}\,[Y_1, Y_2] = [e^{\imath\frac{l_1}{k}\theta}\,Y_1, e^{\imath\frac{l_2}{k}\theta}\,Y_2] = [Y_1, Y_2] \text{ for } 0 \leq \theta \leq 2\pi$$
$$\Longleftrightarrow \frac{l_2}{l_1} = \frac{s}{r} \text{ for some integers } s \text{ and } r \text{ with } |s| < |l_2| \text{ and } |r| < |l_1|.$$

Now let $(l_1, l_2) = p$. Using the fact that the lens spaces $L_k(l_1, p\,l_1)$ and $L_k(1, p)$ are diffeomorphic we conclude that the action is free if and only if $p = 1$ for $Y_1 \neq 0$ and $Y_2 \neq 0$. Hence we now look at the $S^1/\mathbb{Z}_p$ action on $L_k(l_1, l_2)$ and obtain that $S^1/\mathbb{Z}_p$ operates freely on $L_k(l_1, l_2)$ for $Y_1 \neq 0$ and $Y_2 \neq 0$. The action then becomes

$$e^{\imath\theta}\,[Y_1, Y_2] = [e^{\imath\frac{l_1}{p\,k}\theta}\,Y_1, e^{\imath\frac{l_2}{p\,k}\theta}\,Y_2]\,.$$

To compute the fixed point set for the special case of $Y_1 = 0\,, Y_2 \neq 0\,,$ take $\theta = 2\,\pi\,\frac{k\,p}{l_2}$. This leads to the cyclic group $\mathbb{Z}_{l_2} \subset S^1/\mathbb{Z}_p$ which operates trivially on $[0, Y_2] \subset L_k(l_1, l_2)$. Similarly $\mathbb{Z}_{l_1} \subset S^1/\mathbb{Z}_p$ operates trivially on $[Y_1, 0] \subset L_k(l_1, l_2)$. Note that one of these cyclic groups may be trivial, e.g. if $(l_1, l_2) = l_1 = p$. □

Remark 2. If $p = l_1 = l_2$, then both cyclic groups are trivial and the action is free. This is consistent as in this case we get the homogeneous lens space $L_k(p, p)$.

Given a general associated fiber bundle

$$\begin{array}{ccc} F & \longrightarrow & P \times_G F = E \\ & & \big\downarrow \pi \\ & & B \end{array}$$

the tangent bundle of the total space is isomorphic to the direct sum of the pull-back of the tangent bundle of the base space and the vertical bundle ν_E: $\tau_E \cong \pi^*(\tau_B) \oplus \nu_E$. In our case this translates to

$$\tau_{N_{k,l}} \cong \pi^*(\tau_{\mathbb{C}P^2}) \oplus \nu_{N_{k,l}}\,.$$

Proposition 2. *The vertical bundle over $N_{k,l}$ is the direct sum of a two dimensional bundle and a line bundle over $N_{k,l}$: $\nu_{N_{k,l}} \cong F^2 \oplus \mathbb{R}$.*

Proof. Our goal is to show that $\nu_{N_{k,l}}$ splits off a line bundle, i.e. that there is a vector field on $N_{k,l}$. For a general manifold M with a Lie group G action we obtain a vector field on M if there is no point on M which is fixed under the whole group G. Earlier we showed that there is no point in $L_k(l_1,l_2)$ fixed under the whole action of S^1 but now we need the same statement for $N_{k,l}$ itself. Note that we also have a $T^2 = S^1 \times S^1$- action on $S^5 \times L_k(l_1,l_2)$. Hence we get the original action as an $S^1 = T^2/\triangle S^1$- action on $N_{k,l}$, where $\triangle S^1$ stands for the diagonally embedded S^1 . Note that this action is not free, but it does map fibers to fibers as the action is trivial over $\mathbb{C}P^2$. Calculating the fixed point set of the $T^2/\triangle S^1$-action on $N_{k,l}$ we obtain that it coincides with the fixed point set of the T^2- action on $S^5 \times L_k(l_1,l_2)$. But the fixed point set of the T^2- action on $S^5 \times L_k(l_1,l_2)$ is just the fixed point set of the S^1- action on $L_k(l_1,l_2)$ as the action of S^1 on S^5 is the one coming from the Hopf fibration, hence is free. We conclude that the fixed point set of $N_{k,l}$ is finite and hence there exists a vector field on $N_{k,l}$. $\square$

In summary we obtain for the tangent bundle of $N_{k,l}$:

$$\tau_{N_{k,l}} \cong \pi^*(\tau_{\mathbb{C}P^2}) \oplus F^2 \oplus \mathbb{R}\,.$$

This allows us to compute the Pontrjagin and Stiefel-Whitney classes of $N_{k,l}$:

$$\begin{aligned} p_1(\tau_{N_{k,l}}) &= 3\,k^2 + (l_1+l_2)^2 = 3\,k^2 + l_1^2 + l_2^2 \,\in \mathbb{Z}_{l_1 l_2} \\ w_2(\tau_{N_{k,l}}) &= k + l_1 + l_2 \in \mathbb{Z}_2\,. \end{aligned}$$

Here (l_1+l_2) is the Euler class of the two dimensional bundle F^2.

For the generalized Witten manifolds we also use the fact that $\mathbb{C}P^2 \subset \mathbb{C}P^\infty$, to obtain a bundle ξ_k over $E_{l_1\,l_2}$ that pulls back to the stable tangent bundle of $N_{k,l}$. As for the original family $M_{k,l}$ we arrive at the following assumption for the classification.

Condition 2. *Assume that $N_{k,l}$ is homotopy equivalent to $N_{k',l'}$ and ξ_k is isomorphic to $\xi_{k'}$ over the 4-skeleton. Then*

$$\begin{aligned} & (i)\ H^4(N_{k,l}) \cong H^4(N_{k',l'}) \\ & (ii)\ \xi_k \cong \xi_{k'} \text{ over the } 4-\text{skeleton} \\ \Longleftrightarrow\ & (i')\ \mathbb{Z}_{|l_1 l_2|} \cong \mathbb{Z}_{|l_1' l_2'|} \\ & (ii')\ p_1(\xi_k) = p_1(\xi_{k'})\,;\ w_2(\xi_k) = w_2(\xi_{k'}) \\ \Longleftrightarrow\ & (i'')\ |l_1\,l_2| = |l_1'\,l_2'| \\ & (ii'')\ 3\,k^2 + l_1^2 + l_2^2 \equiv 3\,{k'}^2 + {l_1'}^2 + {l_2'}^2 \mod l_1\,l_2\,; \\ & \qquad k + l_1 + l_2 \equiv k' + l_1' + l_2' \mod 2\,. \end{aligned}$$

Again, as we are interested in the stable normal bundle over $E_{l_1 l_2}$ we take the virtual inverse to ξ_k over $E_{l_1 l_2}$ and denote it by ξ_k as well.

3. Cobordism calculation

We use the fact that normal cobordism classes of normal maps $NM(X)$ for

$$\begin{array}{ccc} \nu_M & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{F} & X \end{array}$$

are in one-to-one correspondence with elements of the n-th stable homotopy group of the Thom space of ξ, $\pi_n^S(\mathrm{Th}(\xi))$, which map to $F_*[M]$, modulo self-normal maps. Here $[M] \in H_n(M;\mathbb{Z})$ is the orientation class of M.

$$NM(X) = \left\{\alpha \in \pi_n^S(\mathrm{Th}(\xi)) \text{ such that } H(\alpha) = F_*[M]\right\} / \sim$$

where $H : \pi_n^S(\mathrm{Th}(\xi)) \to H_n(X;\mathbb{Z})$ is the composition of the Thom isomorphism and the Hurewicz map and where $\alpha \sim \beta$ if there exists a bundle map $c : \xi \to \xi'$ such that $c_*(\alpha) = \beta$.

In our case let $h : \pi_7^S(\mathrm{Th}(\xi_k)) \to H_7(\mathrm{Th}(\xi_k);\mathbb{Z})$ be the Hurewicz map and let $H : \pi_7^S(\mathrm{Th}(\xi_k)) \to H_7(E_{l_1 l_2};\mathbb{Z})$ be the composition of h and the Thom isomorphism.

Proposition 3. *Let $\phi : N_{k,l} \to E_{l_1 l_2}$ be the unique lift as in 2.3. Then*

$$Ker(h) \cong \left\{\alpha \in \pi_7^S(Th(\xi_k)) \text{ such that } H(\alpha) = \phi_*[N_{k,l}]\right\}.$$

Proof. Recall that the kernel of h is in one-to-one corespondence with the elements in the inverse image of $\phi_*[N_{k,l}]$, not as group, but the (unnatural) correspondence is being given by subtraction of some specific element in the homotopy group that maps to this homology class. Understood this way, we only need to prove that $\phi_*[N_{k,l}]$ is in the image of H. But just as in the old form of Browder Novikov surgery theory the bundle is equivalent to the normal bundle as a spherical fibre space and the normal bundle of a manifold has a canonical such element. Since $N_{k,l}$ is a manifold the same argument implies that $[N_{k,l}]$ is in the image of H and therefore the same is true for the image of $[N_{k,l}]$ under ϕ_*. □

Remark 3. Proposition 2.1 also holds for the special case of $l_1 = l_2$ which corresponds to the original family of Witten manifolds.

In order to compute the kernel of h we first find the seventh stable homotopy group of the Thom space of the normal bundle of $E_{l_1 l_2}$. As we have detailed knowledge of the cohomology of $E_{l_1 l_2}$ and the dimension of the homotopy group is small, the Adams spectral sequence is the right tool to carry out the calculation. To compute the Ext-groups for the Adams spectral sequence we need to know $H^*(\mathrm{Th}(\xi_k))$ as a module over the Steenrod algebra modulo p where p is a prime number. This calculation consists of the following three steps.

- Find $H^*(E_{l_1 l_2};\mathbb{Z}_p)$ as a module over the Steenrod algebra.

- Use the Thom isomorphism to determine the Steenrod algebra structure of the cohomology $H^*(\mathrm{Th}(\xi_k))$ modulo p.
- Use the Adams spectral sequence to compute $\pi_7^S(\mathrm{Th}(\xi_k)) \otimes \mathbb{Z}_p$.

Remark 4. Note that we only have to consider the primes $p = 2, 3, 5$ as the first homotopy element of order p in the 8- skeleton of $\mathrm{Th}(\xi_k)$ occurs in dimension at least $2p - 3$, see for example [T].

Non-trivial calculations (see Appendix 1) using the above three steps lead to the following results.

- l_1, l_2 odd and $p = 2$.

$$\text{For } k \text{ even}: \pi_7^S(\mathrm{Th}(\xi_k)) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{16}$$
$$\text{For } k \text{ odd}: \pi_7^S(\mathrm{Th}(\xi_k)) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{16}$$

- l_1, l_2 even, k odd and $p = 2$.

$$\pi_7^S(\mathrm{Th}(\xi_k)) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{2^{4+i+j}} \text{ if } 2^i \,||\, l_1 \text{ and } 2^j \,||\, l_2\,.$$

  Here $2^i \,||\, l_1$ stands for 2^i exactly divides l_1, i.e. 2^i is the highest power of 2 dividing l_1.

- $\gcd(3, l_1) = 1$, $\gcd(3, l_2) = 1$ and $p = 3$.

$$\pi_7^S(\mathrm{Th}(\xi_k)) \otimes \mathbb{Z}_3 \cong \begin{cases} \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \text{or} \\ \mathbb{Z}_9 \end{cases}$$

  Here the two possibilities arise from an extension problem.

- $3 \,|\, l_1$ and $3 \,|\, l_2$, $p = 3$.

$$\pi_7^S(\mathrm{Th}(\xi_k)) \otimes \mathbb{Z}_3 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{3^{1+i+j}} \text{ if } 3^i \,||\, l_1 \text{ and } 3^j \,||\, l_2\,.$$

- $\gcd(5, l_1) = 1$, $\gcd(5, l_2) = 1$ and $p = 5$.

$$\pi_7^S(\mathrm{Th}(\xi_k)) \otimes \mathbb{Z}_5 \cong \mathbb{Z}_5$$

- $5 \,|\, l_1$ and $5 \,|\, l_2$, $p = 5$.

$$\pi_7^S(\mathrm{Th}(\xi_k)) \otimes \mathbb{Z}_5 \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{5^{i+j}} \text{ if } 5^i \,||\, l_1 \text{ and } 5^j \,||\, l_2\,.$$

Remark 5. The remaining cases, namely l_1 odd and l_2 even for $p = 2$ as well as $\gcd(3, l_1) = 1$ and $3 \,|\, l_2$ for $p = 3$, lead to different cobordism calculations and will be treated in a sequel to this article.

The analysis of the Hurewicz map given in Appendix 2 leads to the following proposition.

Proposition 4. *If $l_1 = 2^{m_1} 3^{m_2} l_1'$ and $l_2 = 2^{n_1} 3^{n_2} l_2'$ where m_i, n_i for $i = 1, 2$ are nonnegative integers and $(p, l_1') = (p, l_2') = 1$ for $p = 2, 3$, then we obtain for the kernel of the Hurewicz map $h : \pi_7^S(Th(\xi_k)) \to H_7(Th(\xi_k); \mathbb{Z})$:*

$$Ker(h) \otimes \mathbb{Z}_2 \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_{16} & \text{if} \quad m_1 = n_1 = 0\,,\ k \ odd; \\ \mathbb{Z}_4 \oplus \mathbb{Z}_{16} & \text{if} \quad m_1 = n_1 = 0 \ k \ even; \\ \mathbb{Z}_4 \oplus \mathbb{Z}_{16} & \text{if} \quad m_1 > 0\,,\ n_1 > 0\,; \end{cases}$$
$$Ker(h) \otimes \mathbb{Z}_3 \cong \begin{cases} \mathbb{Z}_3 & \text{if} \quad m_2 = n_2 = 0\,; \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \text{if} \quad m_2 > 0\,,\ n_2 > 0\,; \end{cases}$$
$$Ker(h) \otimes \mathbb{Z}_5 \cong \mathbb{Z}_5\,.$$

In order to complete the proof of Theorem 2 we divide the kernel of h by the self-normal maps in $\Omega_7(E_{l_1 l_2}, \xi_k)$. The fibration $O \longrightarrow G \longrightarrow G/O$ leads to a long exact sequence of homotopy groups.

$$\begin{array}{ccccc} \pi_7(O) & \xrightarrow{\alpha} & \pi_7(G) & \longrightarrow & \pi_7(G/O) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \mathbb{Z} & & \pi_7(S^0) \cong \mathbb{Z}_{240} & & 0 \end{array}$$

Hence in order to divide the kernel of h by the self-normal maps we must divide $\Omega_7(E_{l_1 l_2}, \xi_k)$ by $\mathrm{Im}(\alpha) \cong \mathbb{Z}_{240}$ if the 0 - cell is involved in the calculation of $\pi_7^S(\mathrm{Th}(\xi_k))$. Going back to our calculation we consider the sub-complex (as cohomology over the Steenrod algebra) that contains the 0 - cell. We then study the image of the E_2- term of the Adams spectral sequence of S^0 in that sub-complex. In doing so for the various cases we arrive at the following proposition.

Proposition 5. *If $l_1 = 2^{m_1} 3^{m_2} l_1'$ and $l_2 = 2^{n_1} 3^{n_2} l_2'$ where m_i, n_i for $i = 1, 2$ are nonnegative integers and $(p, l_1') = (p, l_2') = 1$ for $p = 2, 3$, then*

$$\Omega_7(E_{l_1 l_2}, \xi_k)/Im(\alpha) \otimes \mathbb{Z}_2 \cong \begin{cases} \mathbb{Z}_2 & \text{if} \quad m_1 = n_1 = 0\,,\ k \ odd; \\ \mathbb{Z}_4 & \text{if} \quad m_1 = n_1 = 0\,,\ k \ even; \\ \mathbb{Z}_4 & \text{if} \quad m_1 > 0\,,\ n_1 > 0\,; \end{cases}$$
$$\Omega_7(E_{l_1 l_2}, \xi_k)/Im(\alpha) \otimes \mathbb{Z}_3 \cong \begin{cases} \mathbb{Z}_3 & \text{if} \quad m_2 > 0\,,\ n_2 > 0\,; \\ 0 & \text{if} \quad m_2 = n_2 = 0\,. \end{cases}$$

In order to obtain $NM_{\mathrm{O}}(E_{l_1 l_2}, \xi_k)$, the group of normal cobordism classes of normal maps, we further add the group $\mathbb{Z}_{28}$. The reason for this is that the cobordism group $\Omega_7(E_{l_1 l_2}, \xi_k)$ does not detect connected sums with homotopy spheres, but there are 28 seven dimensional homotopy spheres. This completes the proof of Theorem 2.

In the case of $l_1\, l_2$ odd we obtain that $H^3(N_{k,l};\mathbb{Z}_2)=0$ and hence general surgery theory implies that know that two such seven dimensional simply connected manifolds N and N' are homeomorphic if and only if $N\#\Sigma^7$ is diffeomorphic to N' for some homotopy sphere Σ^7. In the case of $l_1\, l_2$ even the statement is still correct which follows from [Sm, Proposition 11.3] and is also shown in [KS2, Proposition 2.5]. By the connected sum property of the Eells-Kuiper invariant it follows that $s_1(M\#\Sigma^7)=s_1(M)+\frac{1}{28}$ if Σ^7 is a generator of $\mathbb{Z}_{28}$, the group of homotopy spheres. Thus replacing $s_1(M)$ by $\bar{s}_1(M)=28\, s_1(M)$ yields a homeomorphism classification of the manifolds $N_{k,l}$ as we will see in Sections 4 and 5 below.

4. The invariance groups

For the classification, we employ the invariants $s_i(N_{k,l})$, $i=1,2,3$ mentioned in the introduction. Assume that $M=M^7$ is a closed, smooth, oriented seven dimensional manifold such that M has a compact, smooth, oriented spin coboundary $W=W^8$. Furthermore we require W to satisfy the μ-condition [EK], i.e. the pair (W,M) must have the following properties:

- The homomorphisms in the exact rational cohomology sequence of the pair (W,M), $j^*: H^*(W,M;\mathbb{Q})\longrightarrow H^*(W;\mathbb{Q}))$, are isomorphisms.
- The inclusion homomorphism $i^*: H^1(W;\mathbf{Z}_2)\longrightarrow H^1(M;\mathbf{Z}_2)$ is an epimorphism.

The first condition allows the pull-back of the Pontrjagin classes of W to $H^*(W,M)$. For any such seven dimensional manifold M one defines

$$s_1(W,z)\equiv \frac{1}{2^7\cdot 7}\,\{p_1^2(W)-4\ \mathrm{sign}[W]\}\quad \mathrm{mod}\ 1 \tag{2}$$

which is computed for any orientable spin coboundary W of M satisfying the μ-condition. Here sign$[W]$ stands for the signature of the coboundary W and z is an element of $H^2(W;\mathbb{Z})$ restricting to $u\in H^2(M;\mathbb{Z})$ on the boundary. We also used the simplified notation p_1^2 which stands for $\langle p_1^2,[W,\partial W]\rangle$. Following [KS1] one also defines

$$s_2(W,z)\equiv \frac{1}{2^4\cdot 3}\,\{2\,z^4-z^2\,p_1^2(W)\}\quad \mathrm{mod}\ 1\,; \tag{3}$$

$$s_3(W,z)\equiv \frac{1}{2^2\cdot 3}\,\{8\,z^4-z^2\,p_1^2(W)\}\quad \mathrm{mod}\ 1\,. \tag{4}$$

Note that one can define these invariants in the non-spin case as well, see [KS1] for more details. Here one requires $W=W^8$ to be a smooth manifold with $\partial W=M$ and $z\in H^2(W;\mathbb{Z})$ restricting to $u\in H^2(M;\mathbb{Z})$ on the boundary such that the second Stiefel-Whitney class $w_2(W)=0$ in the spin case and $w_2(W)\equiv z$ mod 2 in the non-spin case. In the non-spin case the invariants become:

$$(5)\quad s_1(W,z) \equiv \frac{1}{2^7\cdot 3\cdot 7}\,\{3\,p_1^2(W) - 12\,\mathrm{sign}[W] - 14\,z^2\,p_1^2(W) + 7\,z^4\} \mod 1\,;$$

$$(6)\quad s_2(W,z) \equiv \frac{1}{2^3\cdot 3}\,\{5\,z^4 - z^2\,p_1^2(W)\} \mod 1\,;$$

$$(7)\quad s_3(W,z) \equiv \frac{1}{2^3}\,\{13\,z^4 - z^2\,p_1^2(W)\} \mod 1\,.$$

We now construct such a coboundary $W^8_{k,l}$ of $N^7_{k,l}$. Recall from (1) that the manifolds $N_{k,l}$ were constructed as orbit spaces of the following S^1- actions.

$$\begin{aligned} \rho_{k,l} : S^1 \times S^5 \times S^3 &\longrightarrow S^5 \times S^3 \\ (z,(u_1,u_2,u_3),(v_1,v_2)) &\longmapsto (z^k\,u_1, z^k\,u_2, z^k\,u_3),(z^{l_1}\,v_1, z^{l_2}\,v_2))\,. \end{aligned}$$

Two diffeomorphisms play a role in the definition of the $N_{k,l}$: the one given by conjugation on $\mathbb{C}^3\times\mathbb{C}^3$ and the one that interchanges the coordinates of $\mathbb{C}^3\times\mathbb{C}^3$. By composing with these diffeomorphisms we can always assume that $k>0$ and $l_1 \geq l_2 > 0$.

First we take a cobordism for $S^5\times S^3$. Here we choose $\mathcal{D} = D^6\times S^3 \subset S^9 = D^6\times S^3 \cup S^5\times D^4$. The action $\rho_{k,l}$ can be extended canonically to the cobordism $\mathcal{D}$ but it will no longer be a free action.

Proposition 6. *Assume $\frac{l_1}{l_2}\notin\mathbb{Z}$. Then the fixed point set of the S^1-action on $\mathcal{D} = D^6\times S^3$ consists of two cyclic groups lying in the circles $(0,(v_1,0))$ and $(0,(0,v_2))$ in $\mathcal{D}$:*

$$Fix(S^1,\mathcal{D}) = \begin{cases} \mathbb{Z}_{l_1} \subset S^1 \textit{ in } (0,(v_1,0)) \subset \mathcal{D} \\ \mathbb{Z}_{l_2} \subset S^1 \textit{ in } (0,(0,v_2)) \subset \mathcal{D} \end{cases}$$

If we remove small equivariant neighborhoods of these orbits, the action becomes free and the quotient will be a smooth eight dimensional manifold $W_{k,l}$ such that $\partial W_{k,l} = N_{k,l}\cup L_1\cup L_2$ where L_1 and L_2 are the seven dimensional lens spaces $L_1 := L_{\frac{l_1}{p}}(k,k,k,\frac{l_2}{p})$ and $L_2 := L_{\frac{l_2}{p}}(k,k,k,\frac{l_1}{p})$ where $(l_1,l_2) = p$.

Remark 6.
- In the case of $\frac{l_1}{l_2}\in\mathbb{Z}$ the fixed point set of the S^1-action on $\mathcal{D}$ becomes $S^3\subset\mathcal{D}$ and hence the above construction of a coboundary $W_{k,l}$ of $N_{k,l}$ fails. Instead we use the description of the manifold $N_{k,l}$ as the associated fiber bundle to the standard Hopf bundle $S^1\longrightarrow S^5\longrightarrow \mathbb{C}P^2$ with the fiber $L_k(l_1,l_2)$ having the following S^1-action. For $z\in S^1$ and $[v_1,v_2]\in L_k(l_1,l_2)$: $z\cdot[v_1,v_2] = [z^{\frac{l_1}{k}}v_1, z^{\frac{l_2}{k}}v_2]$. Hence $N_{k,l} = [S^5\times L_k(l_1,l_2)]/S^1$. The cases of $\frac{l_1}{l_2}\in\mathbb{Z}$ are: $l_1 = l_2$ which corresponds to the original manifolds $M_{k,l}$; and the case of $l_1 = \mu\,l_2$. In both cases there exist smooth four dimensional manifolds $\tilde{L}_{k,l}$ which bound the three dimensional

lens spaces $L_k(l_1, l_2)$. In the homogeneous case of $l_1 = l_2$ one realizes $L_k(l,l)$ as an S^1-bundle over S^2 and constructs the corresponding disk bundle. In the inhomogeneous case $l_1 = \mu\, l_2$ one can use Kirby calculus to construct a corresponding coboundary, see for example [Ro]. In both cases one then forms the eight dimensional manifold $W_{k,l} = [S^5 \times \tilde{L}_{k,l}]/S^1$ which bounds the manifold $N_{k,l}$. Computing the invariants s_i, $i = 1, 2, 3$ in the homogeneous case of $l_1 = l_2$ using this coboundary leads to the results of [KS1].

- Note that for our classification, i.e. for the proof of Theorem 1, it is enough to compute the invariants in the case of $\frac{l_1}{l_2} \notin \mathbb{Z}$ as this case together with the original case of $l_1 = l_2$ produces enough examples of manifolds to generate the full cobordism ring.
- Taking $\mathcal{D} = D^6 \times L_k(l_1, l_2)$, removing small equivariant neighborhoods of the exceptional orbits of the above S^1-action and taking the quotient also produces a bounding manifold for the manifolds $N_{k,l}$. However, this coboundary will no longer be simply connected.

In order to compute the cohomology ring of $W_{k,l}$ we first study the complement of the two disjoint equivariant neighborhoods of the exceptional orbits, i.e. $\tilde{\mathcal{D}} := \mathcal{D}\backslash(\mathfrak{O}_1 \cup \mathfrak{O}_2)$ where $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are the exceptional orbits. Using Lefschetz duality and the Mayer-Vietoris exact sequence we obtain

$$H^i(\tilde{\mathcal{D}}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 3 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 7, 8 \\ 0 & \text{otherwise} \,. \end{cases}$$

Applying the Serre spectral sequence or the Gysin sequence to the fibration $S^1 \longrightarrow \tilde{\mathcal{D}} \longrightarrow W_{k,l}$ we conclude

$$H^i(W_{k,l}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2 \\ \mathbb{Z}_{|l_1 l_2|} & \text{if } i = 4, 6 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 7 \\ 0 & \text{otherwise} \,, \end{cases}$$

where $H^2(W_{k,l}; \mathbb{Z})$ is generated by the first Chern class z of the bundle $S^1 \longrightarrow \tilde{\mathcal{D}} \longrightarrow W_{k,l}$ and $H^4(W_{k,l}; \mathbb{Z})$ is generated by z^2.

In order to compute the invariants $s_i(N_{k,l})$ for $i = 1, 2, 3$ we first determine the first Pontrjagin class of $W_{k,l}$. Recall that the extended action of $\rho_{k,l}$ on $\tilde{\mathcal{D}}$ gives a principal bundle on $\tilde{\mathcal{D}}$, namely $\alpha : S^1 \longrightarrow \tilde{\mathcal{D}} \longrightarrow W_{k,l}$ where $\tilde{\mathcal{D}} \subset D \subset S^9$. Note that the inclusion map $i : \tilde{\mathcal{D}} \hookrightarrow S^9$ is equivariant with respect to the representation

$$\begin{aligned} \delta : S^1 &\longrightarrow U(5) \\ z &\mapsto \begin{pmatrix} z^k & 0 & 0 & 0 & 0 \\ 0 & z^k & 0 & 0 & 0 \\ 0 & 0 & z^k & 0 & 0 \\ 0 & 0 & 0 & z^{l_1} & 0 \\ 0 & 0 & 0 & 0 & z^{l_2} \end{pmatrix} . \end{aligned} \tag{8}$$

In order to determine the tangent bundle of $W_{k,l}$ we follow a construction by Szczarba [Sz] which was also used in [K2]. We denote by ν the normal bundle of $\tilde{\mathcal{D}}$ in S^9. Here ν is of course trivial as $\tilde{\mathcal{D}}$ is a zero codimensional submanifold of S^9. As shown in [Sz, Theorem 1.1] the vector bundles $\tau(W_{k,l}) \oplus \tau_\alpha|S^1 \oplus \nu|S^1$ and $\delta(\alpha) := \tilde{\mathcal{D}} \underset{\delta}{\times} \mathbb{C}^5$ are isomorphic. Here $\tau(W_{k,l})$ stands for the tangent bundle of $W_{k,l}$, $\tau_\alpha|S^1$ is the bundle along the fiber of α restricted to S^1 and $\delta(\alpha)$ is the bundle associated to α by the representation δ. But the tangent bundle along S^1 is trivial and hence $p_1(W_{k,l}) = p_1(\delta(\alpha))$ and

$$\begin{aligned} p_1(W_{k,l}) = p_1(\delta(\alpha)) &= (3\,k^2 + l_1^2 + l_2^2)\; z^2 \,; \\ w_2(W_{k,l}) = w_2(\delta(\alpha)) &= (k + l_1 + l_2)\; z \quad \bmod 2 \,. \end{aligned}$$

Here z is a generator of $H^2(W_{k,l};\mathbb{Z})$. For $i = 1, 2$ let $u_i \in H^2(L_i;\mathbb{Z})$ be the restriction of z to L_i. If $W_{k,l}$ is spin, then it carries a unique spin structure as $H^1(W_{k,l};\mathbb{Z}_2) = 0$. But we also know that

$$H^1(L_i;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \text{ if } 2 \,|\, \dfrac{l_i}{p} \\ 0 \quad \text{otherwise} \,, \end{cases}$$

Hence if $\frac{l_i}{p}$ is odd, L_i also carries a unique spin structure. In the case of even $\frac{l_i}{p}$ one uses the fact that the spin structures of $S^1 \times S^7$ are the pull-back of the spin structures on L_i under the projection $\pi : S^1 \times S^7 \longrightarrow L_i$. But the induced spin structure on L_i just gives the trivial spin structure on $S^1 \times S^7$ in this case. Denote by χ_i the induced spin structure on L_i. If $W_{k,l}$ is non-spin, then $N_{k,l}$ is also non-spin but the lens spaces L_i still carry the spin structures as explained above. Since the invariants are additive with respect to unions we obtain

$$s_j(W_{k,l}, z) = s_j(N_{k,l}, u) + s_j(L_1, u_1, \chi_1) + s_j(L_2, u_2, \chi_2) \quad \text{for} \quad j = 1, 2, 3 \,.$$

Here $u \in H^2(N_{k,l};\mathbb{Z})$ is the restriction of $z \in H^2(W_{k,l};\mathbb{Z})$ to the manifolds $N_{k,l}$.

To complete the calculation of $s_j(W_{k,l}, z)$ we need one more observation.

Lemma 1. *The self-linking number of $N_{k,l}$ is given by*

$$L^{N_{k,l}}(u^2, u^2) = -\frac{1}{l_1\, l_2} \in \mathbb{Q}/\mathbb{Z} \,.$$

Proof. Note that the structure of our proof is similar to that of [K2]. We first use the transgression homomorphism

$$\tau : H^q(S^5 \times S^3;\mathbb{Z}) \longrightarrow H^{q+1}(BS^1;\mathbb{Z})/K_q$$

of the fibration $S^5 \times S^3 \longrightarrow N_{k,l} \longrightarrow BS^1$ together with the homomorphism $\tilde{\pi} : H^q(S^5 \times S^3;\mathbb{Z}) \longrightarrow H^{q-1}(N_{k,l};\mathbb{Z})$ which corresponds to integration along the fiber. Here K_q is the kernel of $H^{q+1}(BS^1;\mathbb{Z}) \longrightarrow H^{q+1}(N_{k,l}, S^5 \times S^3;\mathbb{Z})$. One calculates $\tilde{\pi}(x_3) = l_1\, l_2\, u$ and $\tilde{\pi}(x_5) = u^2$ where $u \in H^2(N_{k,l};\mathbb{Z})$, $x_3 \in H^3(S^5 \times S^3;\mathbb{Z})$ and $x_5 \in H^5(S^5 \times S^3;\mathbb{Z})$ are generators. Using Poincaré duality (PD) we obtain the following commutative diagram.

$$\begin{array}{ccc} H_3(S^5 \times S^3;\mathbb{Z}) & \xrightarrow{\pi_*} & H_3(N_{k,l};\mathbb{Z}) \\ {\scriptstyle PD}\big\uparrow & & {\scriptstyle PD}\big\uparrow \\ H^5(S^5 \times S^3;\mathbb{Z}) & \xrightarrow{\tilde{\pi}} & H^4(N_{k,l};\mathbb{Z}) \end{array}$$

Hence we have that $PD(x_5) = x_3^*$ where x_3^* is the dual to $x_3 \in H^3(S^5 \times S^3;\mathbb{Z})$. From above we know that $\tilde{\pi}(x_5) = u^2$ and hence $L(u^2, u^2) = \langle \beta^{-1}(u^2), \pi_*(x_3^*)\rangle$ where $\beta : H^3(N_{k,l};\mathbb{Q}/\mathbb{Z}) \longrightarrow H^4(N_{k,l};\mathbb{Z})$ is the Bockstein homomorphism. Our goal now is to compute $\beta^{-1}(u^2)$. Note that by construction the Bockstein homomorphism depends only on the 4-skeleton of $N_{k,l}$. In order to describe the 4-skeleton of $N_{k,l}$ recall that $\pi_2(N_{k,l}) \cong \mathbb{Z}$. Hence we choose $\alpha : S^2 \longrightarrow N_{k,l}$ such that $\alpha^*(u)$ is the canonical generator z_2 of $H^2(S^2;\mathbb{Z})$. Using this map α to pull back the S^1-bundle over $N_{k,l}$ we obtain the Hopf fibration $\gamma : S^3 \longrightarrow S^2$ as well as the induced map $\widehat{\alpha} : S^3 \longrightarrow S^5 \times S^3$ such that the following diagram commutes.

$$\begin{array}{ccc} S^1 & & S^1 \\ \big\downarrow & & \big\downarrow \\ S^3 & \xrightarrow{\widehat{\alpha}} & S^5 \times S^3 \\ {\scriptstyle\gamma}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ S^2 & \xrightarrow{\alpha} & N_{k,l} \end{array}$$

Integration along the fiber yields

$$\begin{array}{cccccc} x_3 & H^3(S^5 \times S^3;\mathbb{Z}) & \xrightarrow{\widehat{\alpha}^*} & H^3(S^3;\mathbb{Z}) & l_1 l_2\, z_3 \\ \big\downarrow & {\scriptstyle\tilde{\pi}}\big\downarrow & & {\scriptstyle\tilde{\gamma}}\big\downarrow & \big\downarrow \\ l_1 l_2\, u & H^2(N_{k,l};\mathbb{Z}) & \xrightarrow{\alpha^*} & H^2(S^2;\mathbb{Z}) & l_1 l_2\, z_2 \end{array}$$

Hence $\widehat{\alpha}^*(x_3) = l_1\, l_2\, z_3$ for z_3 the canonical generator of $H^3(S^3;\mathbb{Z})$ and $\alpha_*(\gamma) = \pi_*(\nu)$ in $\pi_3(N_{k,l})$ where $\nu \in \pi_3(S^5 \times S^3)$ is the preimage of x_3^* under the Hurewicz isomorphism. This leads to a description of the 4-skeleton of

$N_{k,l}$ as the map $\alpha \vee \nu : S^2 \vee S^3 \longrightarrow N_{k,l}$ can be extended to a homotopy equivalence up to the 4-skeleton, namely $\alpha \tilde{\vee} \nu : S^2 \vee S^3 \cup e^4 \longrightarrow N_{k,l}$ where e^4 is attached by $\gamma - l_1\, l_2\, e^3$ for e^3 the canonical generator of $\pi_3(S^3)$. We now consider the corresponding chain complex.

$$0 \longrightarrow C_4 \xrightarrow{\ \partial\ } C_3 \xrightarrow{\ \partial\ } C_2 \longrightarrow 0$$

where $e^4 \in C_4$, $e^3 \in C_3$, $e^2 \in C_2$ are the canonical generators. Then the homomorphism

$$\begin{aligned} \rho : C_3 &\longrightarrow \mathbb{Q}/\mathbb{Z} \\ e^3 &\mapsto \frac{1}{l_1\, l_2} \end{aligned}$$

represents a generator of $\mathrm{Hom}(H_3(N_{k,l};\mathbb{Q}/\mathbb{Z})) \cong H^3(N_{k,l};\mathbb{Q}/\mathbb{Z})$. Moreover

$$\begin{aligned} \partial\rho : C_4 \longrightarrow C_3 &\qquad \longrightarrow \mathbb{Q}/\mathbb{Z} \\ e^4 \mapsto -l_1\, l_2\, e^3 &\mapsto -1 \end{aligned}$$

represents the cohomology class $\beta(\rho) = -u^2 \in H^4(N_{k,l};\mathbb{Z})$. But then

$$\begin{aligned} L(u^2, u^2) &= \langle \beta^{-1}(u^2), \pi_*(x_3^*) \rangle \\ &= \langle -\rho, \pi_*(x_3^*) \rangle \\ &= \langle -\rho, e^3 \rangle = -\frac{1}{l_1\, l_2} \end{aligned}$$

which completes the proof of Lemma 1. $\square$

Let $j^* : H^4(W_{k,l}, \partial W_{k,l};\mathbb{Q}) \longrightarrow H^4(W_{k,l};\mathbb{Q})$ be the canonical homomorphism induced by inclusion. Then by [KS1, p. 384] we obtain

$$\begin{aligned} \text{(A)}\ \langle\, z^4, [W_{k,l}, \partial W_{k,l}]\,\rangle &= \langle\, (j^*)^{-1}(z^2) \cup z^2, [W_{k,l}, \partial W_{k,l}]\,\rangle \\ &= L^{\partial W_{k,l}}(z^2 \,|\, \partial W_{k,l}\,,\ z^2 \,|\, \partial W_{k,l}) \\ &= L^{N_{k,l}}(u^2, u^2) + L^{L_1}(u_1^2, u_1^2) + L^{L_2}(u_2^2, u_2^2) \in \mathbb{Q}/\mathbb{Z}\,. \end{aligned}$$

$$\text{(B)}\ \langle\, z^2\, p_1, [W_{k,l}, \partial W_{k,l}]\,\rangle = \langle\, (j^*)^{-1}(z^2) \cup p_1(W_{k,l}), [W_{k,l}, \partial W_{k,l}]\,\rangle\,.$$

As $H^4(L_i;\mathbb{Z}) = \mathbb{Z}_{\frac{l_1}{p}}$ we obtain that $l_i \cdot L^{L_i}(u_i^2, u_i^2) \equiv 0$ in $\mathbb{Q}/\mathbb{Z}$. This implies that

$$\begin{aligned} L^{\partial W_{k,l}}(z^2 \,|\, \partial W_{k,l}\,,\ z^2 \,|\, \partial W_{k,l}) &= -\frac{1}{l_1 l_2} + L^{L_1}(u_1^2, u_1^2) + L^{L_2}(u_2^2, u_2^2) \\ &= \frac{1}{l_1 l_2}\,(-1 + l_1\, l_2\, L^{L_1}(u_1^2, u_1^2) + l_1\, l_2\, L^{L_2}(u_2^2, u_2^2)) \\ &\equiv -\frac{1}{l_1 l_2} \in \mathbb{Q}/\mathbb{Z}\,. \end{aligned}$$

Hence we obtain

$$(9) \qquad \text{(A)}\ \langle z^4, [W_{k,l}, \partial W_{k,l}]\rangle \equiv -\frac{1}{l_1 l_2} \in \mathbb{Q}/\mathbb{Z}$$

$$\text{(B)}\ \langle z^2\, p_1, [W_{k,l}, \partial W_{k,l}]\rangle = \langle (3\,k^2 + l_1^2 + l_2^2)\, z^4, [W_{k,l}, \partial W_{k,l}]\rangle$$

$$(10) \qquad \equiv -\frac{(3\,k^2 + l_1^2 + l_2^2)}{l_1 l_2} \in \mathbb{Q}/\mathbb{Z}$$

Now let $w \in H^4(W_{k,l}, \partial W_{k,l})$ be the generator which is mapped to $-l_1\, l_2\, z^2$ under the canonical homomorphism $H^4(W_{k,l}, \partial W_{k,l}) \longrightarrow H^4(W_{k,l})$. If we choose the orientation on $W_{k,l}$ such that $w\, z^2 = 1$, then $\operatorname{sign}(W_{k,l}) = \operatorname{sign}(-l_1\, l_2) = -|l_1\, l_2|\,$. Hence

$$(11) \qquad \text{(C)}\ \operatorname{sign}(W_{k,l}) = -|l_1\, l_2|\,.$$

In [K2] the invariants were computed for the seven dimensional lens spaces using the description of the invariants in terms of the eta invariants associated to the eigenvalue spectrum of the signature and Dirac operator.

$$s_1(L_p(p_1,p_2,p_3,p_4)) = \frac{1}{2^5 \cdot 7 \cdot p} \sum_{r=1}^{|p|-1} \prod_{j=1}^{4} \cot\left(\frac{r\pi p_j}{p}\right) + \frac{1}{2^4 \cdot p} \sum_{r=1}^{|p|-1} \prod_{j=1}^{4} \frac{1}{\sin\left(\frac{r\pi p_j}{p}\right)};$$

$$s_2(L_p(p_1,p_2,p_3,p_4)) = \frac{1}{2^4 \cdot p} \sum_{r=1}^{|p|-1} (e^{\frac{2\pi\imath r}{|p|}} - 1) \prod_{j=1}^{4} \frac{1}{\sin(\frac{r\pi p_j}{p})};$$

$$(12) \qquad s_3(L_p(p_1,p_2,p_3,p_4)) = \frac{1}{2^4 \cdot p} \sum_{r=1}^{|p|-1} (e^{\frac{4\pi\imath r}{|p|}} - 1) \prod_{j=1}^{4} \frac{1}{\sin(\frac{r\pi p_j}{p})}.$$

Substituting (9), (10) and (11) into the formulas (2) - (7) for the invariants s_i yields:

Proposition 7. *Let $N_{k,l}$ be a generalized Witten manifold with $\frac{l_1}{l_2} \notin \mathbb{Z}$.*

- *If $N_{k,l}$ is spin, then*

$$s_1(N_{k,l}) \equiv \frac{1}{2^7 \cdot 7 \cdot l_1 \cdot l_2}\{4\,|l_1\, l_2| - (3\,k^2 + l_1^2 + l_2^2)^2\} - s_1(L_1) - s_1(L_2) \mod 1;$$

$$s_2(N_{k,l}) \equiv \frac{1}{2^4 \cdot 3 \cdot l_1 \cdot l_2}\{3\,k^2 + l_1^2 + l_2^2 - 2\} - s_2(L_1) - s_2(L_2) \mod 1;$$

$$s_3(N_{k,l}) \equiv \frac{1}{2^4 \cdot 3 \cdot l_1 \cdot l_2}\{3\,k^2 + l_1^2 + l_2^2 - 8\} - s_3(L_1) - s_3(L_2) \mod 1;$$

- *If $N_{k,l}$ is non-spin, then*

$$
\begin{aligned}
s_1(N_{k,l}) &\equiv \frac{1}{2^7 \cdot 3 \cdot 7 \cdot l_1 \cdot l_2}\{12\,|l_1\,l_2| - 3\,(3k^2 + l_1^2 + l_2^2)^2 + 14\,(3\,k^2 + l_1^2 + l_2^2) - 7\} \\
&\quad - s_1(L_1) - s_1(L_2) \mod 1; \\
s_2(N_{k,l}) &\equiv \frac{1}{2^3 \cdot 3 \cdot l_1 \cdot l_2}\{3\,k^2 + l_1^2 + l_2^2 - 5\} - s_2(L_1) - s_2(L_2) \mod 1; \\
s_3(N_{k,l}) &\equiv \frac{1}{2^3 \cdot l_1 \cdot l_2}\{3\,k^2 + l_1^2 + l_2^2 - 13\} - s_3(L_1) - s_3(L_2) \mod 1;
\end{aligned}
$$

Remark 7. Using the second Stiefel-Whitney class we derive the following necessary and sufficient conditions for the existence of a spin structure.

$$
\begin{array}{llllllll}
M_{k,l} & \text{is spin} & \iff & k & \text{even} & & & \\
N_{k,l} & \text{is spin} & \iff & k & \text{even} & & & \\
 & & & \text{or} & k & \text{odd and} & l_1 & \text{odd}\,,\ l_2 \quad \text{even} \\
 & & & \text{or} & k & \text{odd and} & l_1 & \text{even}\,,\ l_2 \quad \text{odd}\,.
\end{array}
$$

Recall that the original Witten manifolds can be described as S^1-bundles over $\mathbb{C}P^2 \times \mathbb{C}P^1$. In this case the corresponding disk-bundle is a zero signature cobordism and the invariants can be computed as follows, see [KS1] for more details.

Proposition 8. *Let $M_{k,l}$ be an Witten manifold and let $m\,k + n\,l = 1$ with n odd and $m \equiv l + 1 \mod 2$ if k is odd.*

- *If $M_{k,l}$ is spin, then*

$$
\begin{aligned}
s_1(M_{k,l}) &\equiv \frac{1}{2^7 \cdot 7 \cdot l^2}\,3\,(l^2 + 3)\,(l^2 - 1)\,k \mod 1; \\
s_2(M_{k,l}) &\equiv \frac{1}{2^4 \cdot 3 \cdot l^2}\,\{-3\,m^2\,k\,(l^2 + 1) + 2\,m\,(l^2 + 3) + 2\,(3\,m\,k - 4)\,m^3\} \mod 1; \\
s_3(M_{k,l}) &\equiv \frac{1}{2^2 \cdot 3 \cdot l^2}\,\{-3\,m^2\,k\,(l^2 + 1) + 2\,m\,(l^2 + 3) + 8\,(3\,m\,k - 4)\,m^3\} \mod 1;
\end{aligned}
$$

- *If $M_{k,l}$ is non-spin, then*

$$
\begin{aligned}
s_1(M_{k,l}) \equiv &\frac{1}{2^7 \cdot 3 \cdot 7 \cdot l^2}\,\{9\,(l^2 + 3)\,(l^2 - 1)\,k - 42\,m^2\,k\,(l^2 + 1) + \\
&28\,m\,(l^2 + 3) + 7\,(3\,m\,k - 4)\,m^3\} \mod 1; \\
s_2(M_{k,l}) \equiv &\frac{1}{2^3 \cdot 3 \cdot l^2}\,\{-3\,m^2\,k\,(l^2 + 1) + 2\,m\,(l^2 + 3) + 5\,(3\,m\,k - 4)\,m^3\} \mod 1; \\
s_3(M_{k,l}) \equiv &\frac{1}{2^3 \cdot l^2}\,\{-3\,m^2\,k\,(l^2 + 1) + 2\,m\,(l^2 + 3) + 15\,(3\,m\,k - 4)\,m^3\} \mod 1;
\end{aligned}
$$

As $s_i(M_{k,l}) \in \mathbb{Z}_r \subset \mathbb{Q}/\mathbb{Z}$ for some positive integer r, we now determine which subgroups $\mathbb{Z}_r$ are generated by the difference $s_i(M_{k,l}) - s_i(M_{k',l})$ for $i = 1, 2, 3$. In order to do so, we first make the following two observations.

- Recall that we can always assume that both k and l are positive integers as we can compose with the diffeomorphism given by conjugation. Hence from now on we assume that $m\,k + n\,l = 1$ with $k > 0$ and $l > 0$.
- In computing $s_i(M_{k,l}) - s_i(M_{k',l}) \in \mathbb{Z}_r$ for $i = 1, 2, 3$ we must include the congruences that come from the corresponding Pontrjagin and Stiefel-Whitney classes as described in Condition 1, i.e. $3\,k^2 \equiv 3\,{k'}^2 \mod l^2$ and $k \equiv k' \mod 2$.

Carefully checking all cases for $s_i(M_{k,l}) - s_i(M_{k',l})$ leads to the following results for the original Witten manifolds.

Proposition 9. *Let $M_{k,l}$ and $M_{k',l}$ be two Witten manifolds such that $3\,k^2 \equiv 3\,{k'}^2 \mod l^2$ and $k \equiv k' \mod 2$. Then*

(1) $\langle s_i(M_{k,l}) - s_i(M_{k',l})\rangle \cong \mathbb{Z}_{2^6}$ *for* $i = 1, 2, 3$ *if* 2^j *exactly divides* l *for some* $j > 0$, *i.e.* $2^j \,||\, l$.

(2) $\langle s_i(M_{k,l}) - s_i(M_{k',l})\rangle \cong \mathbb{Z}_3$ *for* $i = 1, 2, 3$ *if* $3^j \,||\, l$ *where either* k, k' *even or* k, k' *odd.*

(3) $s_i(M_{k,l}) - s_i(M_{k',l}) \equiv 0 \mod 1$ *for* $i = 1, 2, 3$ *if* $p^j \,||\, l$ *for* $p = 5$ *or* $p > 7$ *where either* k, k' *even or* k, k' *odd.*

(4) $\langle s_i(M_{k,l}) - s_i(M_{k',l})\rangle \cong \mathbb{Z}_7$ *for* $i = 1, 2, 3$ *if* $7^j \,||\, l$ *where either* k, k' *even or* k, k' *odd.*

(5) $\langle s_i(M_{k,l}) - s_i(M_{k',l})\rangle \cong \mathbb{Z}_4$ *for* $i = 1, 2, 3$ *if* $gcd(2, l) = 1$ *and* k, k' *odd.*
$\langle s_i(M_{k,l}) - s_i(M_{k',l})\rangle \cong \mathbb{Z}_2$ *for* $i = 1, 2, 3$ *if* $gcd(2, l) = 1$ *and* k, k' *even.*

(6) $s_i(M_{k,l}) - s_i(M_{k',l}) \equiv 0 \mod 1$ *for* $i = 1, 2, 3$ *if* $gcd(p, l) = 1$ *for* $p = 3, 5$ *or* $p > 7$ *where either* k, k' *even or* k, k' *odd.*

(7) $\langle s_i(M_{k,l}) - s_i(M_{k',l})\rangle \cong \mathbb{Z}_7$ *for* $i = 1, 2, 3$ *if* $gcd(7, l) = 1$ *where either* k, k' *even or* k, k' *odd.*

In comparing these groups with the groups $NM_{\mathrm{O}}(E_{l^2})$ of normal cobordisms we find that the groups are the same in all cases except case (5) which corresponds to the case of l odd. Hence we now study the invariants for the generalized Witten manifolds in the case of l_1 and l_2 odd. We start with case of k odd. Note that it is enough to find an element of order 8 in $s_i(N_{k,l}) - s_i(N_{k',l'})$ as we know by the cobordism calculation that the order of the group can be at most 8. But for $l_1 = {l'}_1 = 3$, $l_2 = {l'}_2 = 5$, $k = 1$, $k' = 11$ we obtain that

$$s_1(N_{k,l}) - s_1(N_{k',l'}) = \frac{1}{2^7 \cdot 3 \cdot 7 \cdot l_1 \cdot l_2}\, 2^4 \,\{2^2\,\lambda - 5^2\,l_1\,l_2\} \in \mathbb{Z}_8$$

where $\gcd(2,\lambda) = 1$. In the case of l_1 and l_2 odd and k even we consider the special case of $l_1 = l'_1 = 15$, $l_2 = l'_2 = 17$, $k = 218$, $k' = 292$. Using a computer program to calculate $s_1(L_1)$ and $s_1(L_2)$ we obtain

$$s_1(L_{l_1}(k,k,k,l_2)) + s_1(L_{l_2}(k,k,k,l_1)) - s_1(L_{l_1}(k',k',k',l_2)) - s_1(L_{l_2}(k',k',k',l_1))$$
$$= \frac{1}{2^4 \cdot 7^2 \cdot 11 \cdot 15 \cdot 17 \cdot 10,181}\,(2\,\lambda + 1) \qquad \text{where} \quad 2\,\lambda + 1 = 66,930,799 \quad \text{and}$$

$$\begin{aligned} s_1(N_{k,l}) - s_1(N_{k',l'}) \cong & \frac{1}{2^7 \cdot 7 \cdot l_1 \cdot l_2}\,\{(3k'^2 + l_1^2 + l_2^2)^2 - (3k^2 + l_1^2 + l_2^2)^2\} \\ & - s_1(L'_1) - s_1(L'_2) + s_1(L_1) + s_1(L_2) \\ \cong & \frac{1}{2^7 \cdot 7 \cdot l_1 \cdot l_2}\,\{(9\,(k'^4 - k^4) + 6\,(k'^2 - k^2)\,(l_1^2 + l_2^2)\} \\ & - s_1(L'_1) - s_1(L'_2) + s_1(L_1) + s_1(L_2) \\ \cong & \frac{1}{2^4 \cdot 7^2 \cdot 11 \cdot 15 \cdot 17 \cdot 10,181}\,(2\,\lambda + 1) \in \mathbb{Z}_{16} \end{aligned}$$

This completes our calculation and the group generated by $s_i(N_{k,l}) - s_i(N_{k',l'})$ for $i = 1,2,3$ coincides with the group $NM_{\mathrm{O}}(E_{l_1 l_2})$ of normal cobordisms. In the topological category one uses the invariants $\bar{s}_i$, $i = 1,2,3$ to obtain the analogous result for $NM_{\mathrm{TOP}}(E_{l_1 l_2})$. Hence Theorem 4 follows.

5. The surgery problem

Let $N_{k,l}$ and $N_{k',l'}$ be elements of the same normal cobordism class of normal maps. Then we obtain the following surgery problem.

$$\begin{array}{ccc} N^7_{k',l'} & \xrightarrow{h_1} & E_{l_1 l_2} \\ {\scriptstyle i_1}\big\downarrow & & \\ W^8 & \xrightarrow{h} & E_{l_1 l_2} \times I \\ {\scriptstyle i_0}\big\uparrow & & \\ N^7_{k,l} & \xrightarrow{h_0} & E_{l_1 l_2} \end{array}$$

where i_0 and i_1 are the natural inclusions, h_0 and h_1 are homotopy equivalences up the the 4-skeleton and W^8 is a cobordism for $N_{k,l}$ and $N_{k',l'}$. Recall that $l' = l'_1\,l'_2 = l_1\,l_2 = l$ is a necessary condition (condition 2). First note that by standard surgery arguments we can make h into an isomorphism on homology up to and including dimension 3.

Our goal is to show that we can use surgery to turn W into an h-cobordism. The h-cobordism theorem then implies that $N_{k,l}$ and $N_{k',l'}$ are diffeomorphic which completes the proof of Theorem 3.

To simplify the notation denote $E_{l_1 l_2}$ by X, $N_{k,l}$ and $N_{k',l'}$ by N_0 and N_1 respectively. We first use the following long exact sequence.

$$\ldots \longrightarrow H_i(X) \longrightarrow H_i(h) \longrightarrow H_{i-1}(W) \longrightarrow H_{i-1}(X) \longrightarrow \ldots$$

Here all homology groups are taken with coefficients in $\mathbb{Z}$ and $H_i(h)$ is defined via the homology of the mapping cylinder of h. As h is an isomorphism on homology up to and including dimension 3, exactness of this sequence implies that

$$H_i(h) \cong 0 \text{ for all } i \leq 4\,.$$

By Hurewicz this implies that $\pi_5(h) \cong H_5(h)$ and we obtain the following diagram.

$$\begin{tikzcd}
\ldots \arrow[r] & \pi_5(X) \arrow[r] & \pi_5(h) \arrow[r] \arrow[d, "\cong"'] & \pi_4(W) \arrow[r] & \pi_4(X) \arrow[r] & \ldots \\
\ldots \arrow[r] & H_5(X) \arrow[r] & H_5(h) \arrow[r] & H_4(W) \arrow[r] & H_4(X) \cong 0 \arrow[r] & \ldots
\end{tikzcd}$$

Recall that the elements of the group $\pi_{n+1}(h)$ are defined by commutative diagrams

$$\begin{tikzcd}
S^n \arrow[r, "k"] \arrow[d, "\alpha"'] & D^{n+1} \arrow[d, "\beta"'] \\
W \arrow[r, "h"] & X
\end{tikzcd}$$

where k is inclusion of the boundary and all maps and homotopies are base point preserving (see [H]). Thus β defines a map $\bar{h} : W \cup_\alpha D^{n+1} \longrightarrow X$ extending h.

Our goal is to define a unimodular intersection form on $H_4(W)$ even though the model space X is not a Poincaré duality space. This idea of extending the surgery problem to a non-Poincaré duality model space by using the fact that the middle dimensional homology is zero has been used in a different context by [CS].

Let $u \in H_4(W)$. As $H_5(h) \longrightarrow H_4(W)$ is surjective there exists some element $\phi \in H_5(h)$ that gets mapped to u. As $\pi_5(h) \cong H_5(h)$ there exists a corresponding element in $\pi_5(h)$, i.e.

$$\begin{tikzcd}
S^4 \arrow[r, "k"] \arrow[d, "\alpha"'] & D^5 \arrow[d, "\beta"'] \\
W \arrow[r, "h"] & X
\end{tikzcd}$$

which induces a framed immersion of S^4 into W. We can now use the standard intersection form for those framed immersions. What remains to show is that this intersection form actually lies in the surgery obstruction group, i.e. we need to show that $H_4(W)$ is torsion free and that the intersection form is unimodular. Recall that the intersection form is a quadratic form $H_4(W) \times H_4(W) \longrightarrow \mathbb{Z}$ and that the adjoint of the intersection pairing, namely $H_4(W) \longrightarrow \mathrm{Hom}(H_4(W), \mathbb{Z})$ is given by Poincaré duality. If we can show that this adjoint is an isomorphism, then the intersection form is unimodular and $H_4(W)$ is torsion-free. Note that we have the following long exact sequence of the pair (W, N_i) for $i = 0, 1$.

$$\dots \longrightarrow H_4(N_i) \longrightarrow H_4(W) \longrightarrow H_4(W, N_i) \longrightarrow H_3(N_i) \longrightarrow H_3(W) \longrightarrow \dots$$

As $H_4(N_i) = 0$ and $H_3(N_i) \cong H_3(W)$ this implies that $H_4(W)$ is isomorphic to $H_4(W, N_i)$ for $i = 0, 1$. Hence we obtain the following for the adjoint of the intersection form:

$$H_4(W) \cong H_4(W, N_0) \cong_{PD} H^4(W, N_1) \cong \mathrm{Hom}(H_4(W, N_1), \mathbb{Z}) \cong \mathrm{Hom}(H_4(W), \mathbb{Z}).$$

But this implies that the adjoint of the intersection form is an isomorphism and we now have a unimodular intersection form on $H_4(W)$, i.e. we have an element in the surgery obstruction group $L_4(e)$.

Now suppose that after picking the correct basis for $H_4(W)$ this intersection form looks like $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Then we can apply standard surgery methods on embedded 4-spheres to turn W into an h-cobordism. Now if our intersection form does not look like $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, i.e. is not trivial, then we can always construct a new cobordism W such that the new intersection form is trivial. In fact, we can use the realization theorem here, namely that every intersection form can be realized by a cobordism W, for example through plumbing [KM]. Assume that our cobordism W has some intersection form $A \neq \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Using the realization theorem we construct a cobordism $\bar{W}$ that has $-A$ as intersection form and that has some homotopy 7-sphere as boundary. We then form the connected sum $W \# \bar{W}$. This new cobordism has $\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$ as intersection form, which is equivalent to $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, and we have thus reduced this case to the trivial one. However, we now obtain for the boundary of this new cobordism $\partial(W \# \bar{W}) = N_0 \cup N_1 \cup \Sigma^7$. But the homotopy sphere Σ^7 does not contribute in the topological case and can by detected by the Eells-Kuiper invariant in the smooth case. Hence we have reduced this case to the trivial case, which thus completes the proof of Theorem 3.

6. Appendix 1: Calculating $\boldsymbol{\pi_7^S}(\mathbf{Th}(\xi_\mathbf{k})) \otimes \mathbb{Z}_\mathbf{p}$

We first complete the calculation for the case of $l_1 = l_2 = l$. We will see that the general case follows directly from these calculations. Recall that we first need to find $H^*(E_{l^2}; \mathbb{Z}_p)$ as a module over the Steenrod algebra $\mathcal{A}_p$.

We use the fibration $K(\mathbb{Z}, 3) \longrightarrow E_{l^2} \longrightarrow \mathbb{C}P^\infty$. Then the E_2-term of the Serre spectral sequence for $H^*(E_{l^2}; \mathbb{Z}_p)$ becomes

$$E_2^{s,t} = H^s(\mathbb{C}P^\infty) \otimes H^t(K(\mathbb{Z}, 3)).$$

By a theorem of Serre [S] one knows that $H^*(K(\mathbb{Z},n),\mathbb{Z}_p)$ is a free, graded, commutative algebra on free generators $s_q^I i_n$ where i_n is in $H^n(K(\mathbb{Z},n),\mathbb{Z}_p)$ and I is an admissible sequence, $I=(a_1,\ldots,a_m)$ with a_m even and excess $e(I)<n$. Hence $E_2^{s,t}=\mathbb{Z}_2[x]\otimes\mathbb{Z}_2[i_3,s_q^2 i_3,s_q^4 s_q^2 i_3]$ for $p=2$ and cohomological dimension less than or equal to 10. Note that we are interested in the seventh stable homotopy group, hence in all of our calculations we restrict ourselves to dimensions less than or equal to 10. For p odd we obtain

$$E_2^{s,t}=\mathbb{Z}_p[x]\otimes\mathbb{Z}_p[i_3,P^1 i_3,\beta P^1 i_3]/(i_3^2,(P^1 i_3)^2\,,$$

where $P^i : H^k(X;\mathbb{Z}_p)\longrightarrow H^{k+2i(p-1)}(X;\mathbb{Z}_p)$ is a natural transformation of functors and a homomorphism that is used in the axiomatic construction of the algebra $\mathcal{A}_p$, and $\beta : H^k(X;\mathbb{Z}_p)\longrightarrow H^{k+1}(X;\mathbb{Z}_p)$ is the Bockstein coboundary operator associated with the exact coefficient sequence $0\longrightarrow\mathbb{Z}_p\longrightarrow\mathbb{Z}_{p^2}\longrightarrow\mathbb{Z}_p\longrightarrow 0$.

We now divide the calculation into cases according to the prime p used and the parity of k and l.

6.1. **p = 2, l odd, any k.** We examine the Serre spectral sequence of $K(\mathbb{Z},3)\longrightarrow E_{l^2}\longrightarrow\mathbb{C}P^\infty$ for $s+t\le 10$. Note that $H^*(\mathbb{C}P^\infty)$ has basis $\{1,x,x^2,x^3,x^4,x^5\}$ in dimensions $\{0,2,4,6,8,10\}$ and $H^*(K(\mathbb{Z},3))$ has basis $\{1,i_3,s_q^2 i_3=u,s_q^3 i_3=i_3^2=s_q^1 s_q^2 i_3=s_q^1 u,i_3 u,i_3 s_q^1 u,u^2\}$ in dimensions $\{0,3,5,6,8,9,10\}$. Note that if the dimension of X is equal to n, then $s_q^n=x^2$, hence $s_q^3 i_3=i_3^2$.

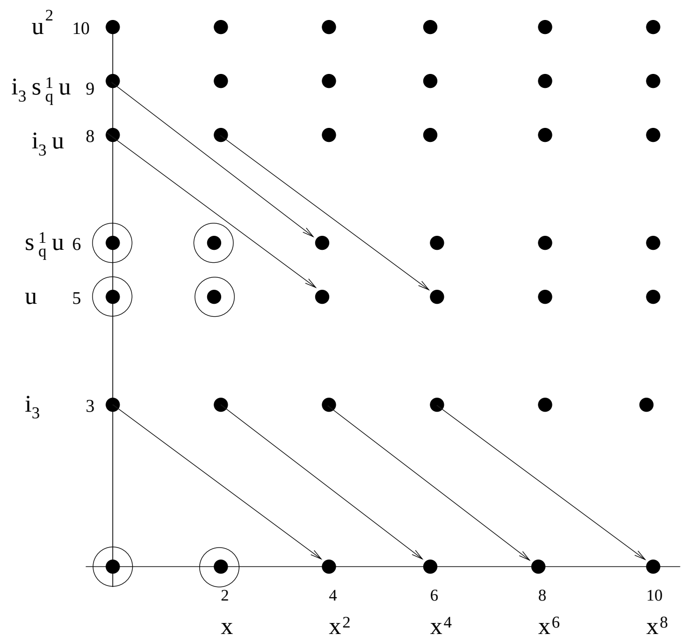

Figure 1. The spectral sequence for E_{l^2}

As we are in the case of $p = 2$ and l odd, we obtain that the generator x^2 of the cohomology of $\mathbb{C}P^\infty$ goes to 0 in the cohomology of E_{l^2} (recall that $H^4(E_{l^2}, \mathbb{Z}) \cong \mathbb{Z}_{l^2}$). Hence x^2 does not survive in the Serre spectral sequence, which means that $d_4(i_3) = x^2$. Note that for dimensional reasons x^2 cannot be in the image of any other differential. Hence we obtain that $d_4(i_3) = x^2$ and $d_4(i_3 \otimes x^j) = x^{j+2}$. Also note that in this case $i_3 \in H^3(K(\mathbb{Z}, 3))$ is transgressive, hence $s_q^2 i_3$ is transgressive as well. This implies that $d_6(s_q^2 i_3)$ is defined. Moreover if $y \in d_4(i_3)$, then $s_q^i(y) \in d_n^{0,n-1}(s_q^i i_3)$. Hence $s_q^2(d_4(i_3)) = d_6(s_q^2 i_3)$ and $d_6(u) = d_6(s_q^2 i_3) = s_q^2(d_4(i_3)) = s_q^2(x^2) = 2\, s_q^2 x + (s_q^1 x)^2 = 0$. We conclude that u is a permanent cycle and that $s_q^1 u = i_3^2$. Also $u \otimes x^j$ and $i_3^2 \otimes x^j$ are permanent cycles. By the multiplicative properties of the spectral sequence we obtain that $d_4(i_3 u) = u \otimes x^2$; $d_4(i_3 u \otimes x^j) = u \otimes x^{j+2}$ and $d_4(i_3^3) = i_3^2 \otimes x^2$. The only remaining generator is u^2 but $d_i(u^2) = 2\, d_i(u) = 0$, so u^2 is a permanent cycle as well. Hence for $s+t \leq 10$ we are left with the following as a basis for $H^*(E_{l^2})$: $\{1 \otimes 1, 1 \otimes x, u \otimes 1, i_3^2 \otimes 1, u \otimes x, i_3^2 \otimes x, u^2 \otimes 1\}$ in bi-degrees $\{(0, 0), (2, 0), (0, 5), (0, 6), (2, 5), (2, 6), (0, 10)\}$.

We are now ready to determine the Steenrod algebra structure. There is one class in dimension 0: $1 \otimes 1$; and no operations by the "unstable" axiom for the cohomology of a space. There is one class, $1 \otimes x$, in dimension 2 and no operations as this class can only support a s_q^1 or s_q^2 but there are no classes in dimensions 3 and 4. Moreover, there is one class in dimension 5, $u \otimes 1$, with $s_q^1 u = i_3^2$ and $s_q^2 u = 0$ as this is already the case in $K(\mathbb{Z}, 3)$): $s_q^2 u = s_q^2(s_q^2 i_3) = s_q^3(s_q^1 i_3) = 0$. Also $s_q^4(u) = 0$ as $H^7(K(\mathbb{Z}, 3)) = 0$. There is one class in dimension 6, $i_3^2 \otimes 1$, with no s_q^1 as $(s_q^1)^2 = 0$ and with no s_q^2 as there is no s_q^2 in $K(\mathbb{Z}, 3)$: $s_q^2(i_3^2) = i_3 s_q^2 i_3 + s_q^1 i_3 \cdot s_q^1 i_3 + s_q^2 i_3 \cdot i_3 = 2 \cdot i_3 u + (s_q^1 i_3)^2 = 0$. Also $s_q^4(i_3^2) = u^2$ as $s_q^4(i_3^2) = i_3 \cdot s_q^4 i_3 + s_q^1 i_3 \cdot s_q^1 i_3 + \ldots + s_q^4 i_3 \cdot i_3 = s_q^2 i_3 \cdot s_q^2 i_3 = u^2$. There is one class in dimension 7, $u \otimes x$, with $s_q^1(u \otimes x) = i_3^2 \otimes x$ as in the E_2-term of the spectral sequence $s_q^1(u \otimes x) = u \otimes s_q^1 x + s_q^1 u \otimes x = s_q^1 u \otimes x = i_3^2 \otimes x$. As there are no group extensions, this is true in $H^*(E_{l^2})$ as well. Also $s_q^2(u \otimes x) = 0$ as $H^9(E_{l^2}) = 0$ and there is no s_q^4 for dimensional reasons. There is one class in dimension 8, $i_3^2 \otimes x$, with no s_q^1 as $(s_q^1)^2 = 0$. Also $s_q^2(i_3^2 \otimes x) = 0$ as the only element in dimension 10 is in a lower fibration. Again there are no higher s_q^i out of dimensional reasons. Lastly, there is one class, $u^2 \otimes 1$, in dimension 10 with no operations out of dimensional reasons. In summary, we obtain for $H^*(E_{l^2})$ as an $\mathcal{A}_p$-module in the case of $p = 2$ and l odd:

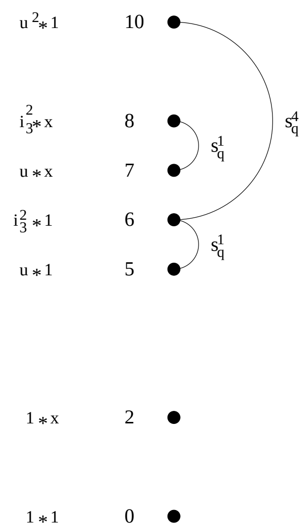

Figure 2. $H^*(E_{l^2})$ for l odd

We now use the Thom isomorphism to determine the Steenrod algebra structure of $H^*(Th(\xi_k))$ modulo p. In order to do so we must consider subcases according to the parity of k.

6.1.1. *k even, l odd, $p = 2$.* In this case $w_i(\xi_k) = 0$ for all $i > 0$, so $s_q^i(\mathcal{U}) = \mathcal{U} \cup w_i(\xi_k) = 0$ where $\mathcal{U}$ is the Thom class in $H^0(Th(\xi_k))$. Hence the $\mathcal{A}_2$-module structure of $H^*(Th(\xi_k))$ is the same as the one of $H^*(E_{l^2})$.

6.1.2. *k odd, l odd, $p = 2$.* We first label the elements of $H^*(Th(\xi_k))$ by $t(y)$ where $y \in H^*(E_{l^2})$. Recall the Thom isomorphism

$$\begin{aligned} t : H^*(E_{l^2}) &\longrightarrow \bar{H}^*(Th(\xi_k)) \\ x &\mapsto \mathcal{U} \cup x = t(x)\,. \end{aligned}$$

Here $w_2(\xi_k) = x, w_8(\xi_k) = x$ and all other Stiefel-Whitney classes are zero. Note that $s_q^8\mathcal{U} = \mathcal{U} \cup w_8(\xi_k) = \mathcal{U} \cup x^4 = t(x^4)$ but there is no $t(x^4)$ in dimension eight as the generator in dimension eight is $i_3^2 \otimes x$. Therefore we obtain that $s_q^2\mathcal{U} = \mathcal{U} \cup w_2(\xi_k) = \mathcal{U} \cup x = t(x)$ and $s_q^i\mathcal{U} = 0$ for all $i > 0, i \neq 2$. More generally we obtain the following formula.

Lemma 2. $s_q^i(t(y)) = t\left(\sum_{j=0}^{i} w_j(\xi_k) \cup s_q^{i-j}(y)\right)$.

Proof.

$$s_q^i(t(y)) = s_q^i(\mathcal{U} \cup y) = \sum_{j=0}^{i} s_q^j\mathcal{U} \cup s_q^{i-j}(y) = \sum_{j=0}^{i} \mathcal{U} \cup w_j(\xi_k) \cup s_q^{i-j}(y)$$

$$= \mathcal{U} \cup \sum_{j=0}^{i} w_j(\xi_k) \cup s_q^{i-j}(y) = t\left(\sum_{j=0}^{i} w_j(\xi_k) \cup s_q^{i-j}(y)\right)$$

since $s_q^j \mathcal{U} = \mathcal{U} \cup w_j(\xi_k)$. □

Using the basis previously developed for E_{l^2} and the above formula for $s_q^i(t(y))$ we obtain one class, $t(1)$, in dimension 0 and find that

$$\begin{aligned} s_q^i(t(1)) &= t\left(x \cup s_q^{i-2}(1) + s_q^i(1)\right) \\ &= \begin{cases} 0, & \text{for all } i > 0, i \neq 2 \text{ and} \\ t(x) & \text{for } i = 2 \end{cases} \end{aligned}$$

Similarly there is one class, $t(x)$, in dimension 2 without any operations. There is one class, $t(u)$, in dimension 5 and $s_q^1(t(u)) = t\left(s_q^1 u\right) = t\left(i_3^2\right)$ and $s_q^2(t(u)) = t(u \otimes x)$ as $s_q^0(u) = u$. In dimension 6 there is one class $t(i_3^2)$ and $s_q^1(t\left(i_3^2\right)) = 0$, $s_q^2(t\left(i_3^2\right)) = t(i_3^2 \otimes x)$, $s_q^4(t\left(i_3^2\right)) = t(u^2)$ as $s_q^0(i_3^2) = i_3^2$ and $s_q^4(i_3^2) = u^2$. There is one class $t(u \otimes x)$ in dimension 7 with $s_q^1(t(u \otimes x)) = t\left(i_3^2 \otimes x\right)$. In dimension 8 there is one class, $t\left(i_3^2 \otimes x\right)$, with no operations as there are no operations on $i_3^2 \otimes x$. Lastly, there is one class, $t(u^2)$, in dimension 10 with no operations as there are no operations on u^2. Therefore we obtain for the Steenrod algebra structure of $H^*(Th(\xi_k))$:

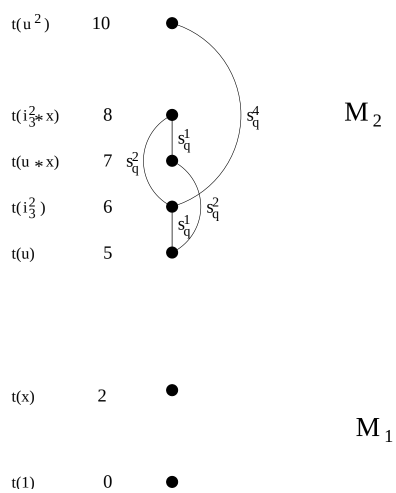

Figure 3. $H^*(Th(\xi_k))$ for l odd

Note that this picture is disconnected, hence the cohomology splits into two modules $M_1 \oplus M_2$. This follows from the fact that there are no operations between M_1 and M_2.

In the last step we use the Adams spectral sequence to compute $\pi_7^S(Th(\xi_k)) \otimes \mathbb{Z}_p$. Recall that we are still in the case of $p = 2$ and l odd. We start our calculation for the subcase 2 (k odd). Since the cohomology splits into two modules

$M_1 \oplus M_2$ in this case, the functorality of Ext implies that $\mathrm{Ext}^{s,t}(M_1 \oplus M_2; \mathbb{Z}_2) = \mathrm{Ext}^{s,t}(M_1; \mathbb{Z}_2) \oplus \mathrm{Ext}^{s,t}(M_2; \mathbb{Z}_2)$. Note that the module M_1 fits into the short exact sequence $0 \longrightarrow \Sigma^2\mathbb{Z}_2 \longrightarrow M_1 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$. This is because we have $S^0 \longrightarrow \Sigma^{-2}\mathbb{C}P^2 \longrightarrow S^2$. Using the corresponding long exact Ext-sequence we obtain

$$\longrightarrow \mathrm{Ext}^{s-1,t}(\Sigma^2\mathbb{Z}_2) \xrightarrow{h_1} \mathrm{Ext}^{s,t}(\mathbb{Z}_2) \longrightarrow \mathrm{Ext}^{s,t}(M_1) \longrightarrow \mathrm{Ext}^{s,t}(\Sigma^2\mathbb{Z}_2)$$
$$\xrightarrow{h_1} \quad \mathrm{Ext}^{s+1,t}(\mathbb{Z}_2) \quad \longrightarrow$$

This leads to the following exact sequence:

$$0 \longrightarrow \mathrm{coker}(h_1 : \mathrm{Ext}^{s,t}(\mathbb{Z}_2)) \longrightarrow \mathrm{Ext}^{s,t}(M_1) \longrightarrow \mathrm{Ker}(h_1 : \mathrm{Ext}^{s,t}(\Sigma^2\mathbb{Z}_2)) \longrightarrow 0\,.$$

Now we use the table for $\mathrm{Ext}_{\mathcal{A}_2}(\mathbb{Z}_2, \mathbb{Z}_2)$, see for example [R, p.381], and compute the cokernel and kernel of h_1. The results can be summarized in the following diagram.

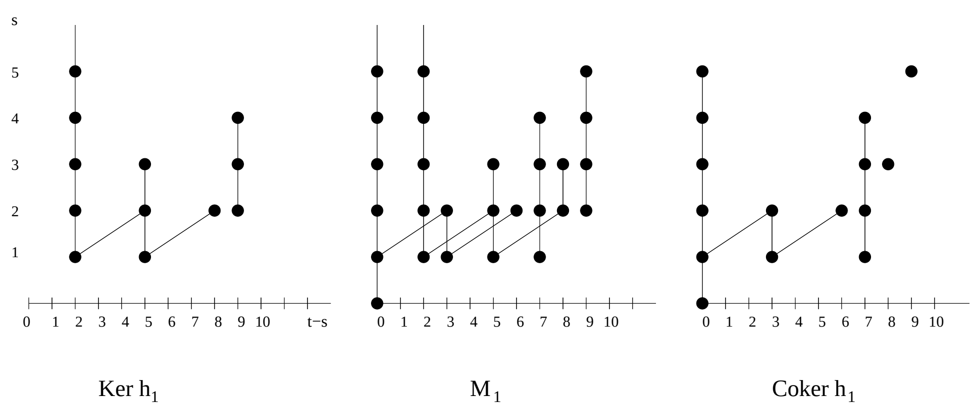

Figure 4. $\mathrm{Ext}^{s,t}(M_1; \mathbb{Z}_2)$

Recall that the module M_2 has four generators, one each in dimensions $5, 6, 7, 8$. Note that in this case the structure over the Steenrod algebra can be described by the statement that s_q^1 and s_q^2 are as non-trivial as possible. We now must first find a minimal projective resolution of M_2, i.e. we find an exact sequence of projective modules $\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M_2$. Recall that we can take as P_0 the free module with generators the elements of M_2, i.e. we can take $\mathcal{A}_2$ itself as P_0. Using the basic Adem relations [MT] we obtain the following diagram for M_2.

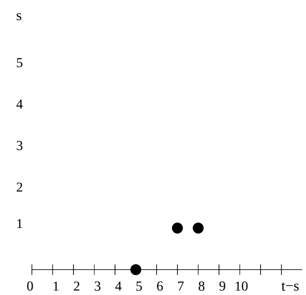

Figure 5. $\mathrm{Ext}^{s,t}(M_2;\mathbb{Z}_2)$

Claim 1. *There are no differentials in the E_2-term of the Adams spectral sequence in the case of l odd and k odd.*

Proof. By looking at the combined diagram for M_1 and M_2 we obtain three possibilities for differentials, d_2 from the element $(8,1)$, d_3 from $(8,1)$ and d_2 from $(8,2)$. Here d_2 from $(8,1)$ must be zero by h_0-linearity and d_2 from $(8,2)$ must be zero by h_2-linearity (e.g. $(8,2) = h_2[(5,1)]$ implies that $d_2[(8,2)] = d_2[h_2[(5,1)]] = d_2(h_2)\,(5,1) + h_2\,d_2[(5,1)] = h_2\,d_2[(5,1)] = 0$). Now note that h_2 and $(5,0)$ are permanent cycles in the Adams spectral sequence, hence $(8,1) = h_2 \cdot (5,0)$ is a permanent cycle as well. But this implies that d_3 from $(8,1)$ must be zero as well. $\square$

Conclusion 1. $\pi_7^S(E_{l^2};\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{16}$ *for l odd and k odd.*

Claim 2. *There are no differentials in the E_2-term of the Adams spectral sequence in the case of l odd and k even.*

Proof. In the case of l odd and k even we use the $\mathcal{A}_2$-module structure of $H^*(Th(\xi_k))$ to conclude that $\mathrm{Ext}_{\mathcal{A}_2}(M,\mathbb{Z}_2) = \mathrm{Ext}_{\mathcal{A}_2}(M_1,\mathbb{Z}_2) \oplus \mathrm{Ext}_{\mathcal{A}_2}(M_2,\mathbb{Z}_2) \oplus \mathrm{Ext}_{\mathcal{A}_2}(M_3,\mathbb{Z}_2) \oplus \mathrm{Ext}_{\mathcal{A}_2}(M_4,\mathbb{Z}_2)$ where M_1 has one class in dimension zero and no operations; M_2 has one class in dimension two and no operations; M_3 has one class each in dimensions $5, 6$ and 10 and a s_q^1 from the class in dimension 5 to the one in dimension 6 as well as a s_q^4 from the class in dimension 6 to the one in dimension 10; M_4 has one class each in dimensions 7 and 8 and a s_q^1 from the class in dimension 7 to the one in dimension 8. Hence $\mathrm{Ext}(M_1,\mathbb{Z}_2) = \mathrm{Ext}(M_2,\mathbb{Z}_2) = \mathrm{Ext}(\mathcal{A}_2,\mathbb{Z}_2)$ and as before we use the short exact sequence for M_3 and M_4, e.g. $0 \longrightarrow \Sigma^1\mathbb{Z}_2 \longrightarrow M_3 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$ and

$$0 \longrightarrow \mathrm{coker}(h_0 : \mathrm{Ext}^{s,t}(\mathbb{Z}_2)) \longrightarrow \mathrm{Ext}^{s,t}(M_3) \longrightarrow \mathrm{Ker}(h_0 : \mathrm{Ext}^{s,t}(\Sigma\mathbb{Z}_2)) \longrightarrow 0\,.$$

Putting all four pieces together we obtain nine possible differentials in the Adams spectral sequence. All but two of them must be zero by h_0-, h_1-, h_2-linearity arguments. The two remaining and so far unresolved differentials are $d_2[(7,0)]$ and $d_2[(5,0)]$. For those we recall that ξ_k is a bundle over E_{l^2} and E_{l^2} is a bundle over $\mathbb{C}P^\infty$. The bundle map induced by the projection map from E_{l^2} to $\mathbb{C}P^\infty$

induces a map on Ext in analogously to the way we obtain induced maps on cohomology. Recall that $\mathrm{Ext}_{\mathcal{A}_2}(Th(\mathbb{C}P^\infty), \mathbb{Z}_2) = \mathrm{Ext}_{\mathcal{A}_2}(N_1, \mathbb{Z}_2) \oplus \mathrm{Ext}_{\mathcal{A}_2}(N_2, \mathbb{Z}_2)$ where N_1 has one class in dimension zero and no operations and N_2 has one class in dimensions $2, 3, 6, 7$ and 10 and a s_q^1 from the class in dimension 2 to the one in dimension 3, a s_q^1 from the class in dimension 6 to the one in dimension 7 as well as a s_q^4 from the class in dimension 3 to the one in dimension 7 and a s_q^4 from the class in dimension 6 to the one in dimension 10. The induced map π_{Ext} from $\mathrm{Ext}_{\mathcal{A}_2}(Th(E_{l^2})), \mathbb{Z}_2)$ to $\mathrm{Ext}_{\mathcal{A}_2}(Th(\mathbb{C}P^\infty), \mathbb{Z}_2)$ must map the class in dimension 0 to the one in dimension 0 by an isomorphism and it must map M_3 and M_4 to zero. Now if $d_2[(0,7)] = (6,2)$ then naturality of the Adams spectral sequence implies that $d_2[\pi_{\mathrm{Ext}}((0,7))] = \pi_{\mathrm{Ext}}((6,2)) \neq 0$ which is a contradiction since $\pi_{\mathrm{Ext}}((0,7)) = 0$. Hence $d_2[(0,7)] = 0$.

In order to show that $(5,0)$ is a permanent cycle we again use the naturality of the Adams spectral sequence. We now map $\mathrm{Ext}_{\mathcal{A}_2}(Th(E_{l^2})), \mathbb{Z}_2)$ to the Ext-group over $\mathcal{A}_2$ of an $\mathcal{A}_2$-module N which differs from the one of E_{l^2} by the lack of the class in dimension two. Let $x = (8,1), y = (7,4)$ and $\bar{x}$ and $\bar{y}$ the image of x and y under this map. Assume that $d_3(x) = y$. By h_0-linearity we know that $d_2(x) = d_2(\bar{x}) = 0$. For $x, \bar{x}$ and $y, \bar{y}$ in the E_3-term the map induced on the spectral sequence will send x to $\bar{x}$ and y to $\bar{y}$. Hence if $d_3(x) = \lambda y$, then $d_3(\bar{x}) = \lambda \bar{y}$. But $d_3(\bar{x}) = 0$, as $(5,0)$ in $\mathrm{Ext}_{\mathcal{A}_2}(N)$ is a permanent cycle. Note that $d_3(\bar{x}) = 0$ if $(5,0)$ is a permanent cycle by h_2-linearity. But then $\lambda = 0$ and $d_3(x) = 0$. Hence again by h_2-linearity $(5,0)$ must be a permanent cycle. □

Conclusion 2. *$\pi_7^S(Th(\xi_k, \mathbb{Z}_2)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{16}$ for l odd and k even.*

6.2. **$\mathbf{p = 2}$ and $\mathbf{l}$ even.** Again we examine the Serre spectral sequence for

$$K(\mathbb{Z}, 3) \longrightarrow E_{l^2} \longrightarrow \mathbb{C}P^\infty$$

and $s + t \leq 10$. As l is even, $l^2 x^2$ already vanishes in the cohomology modulo 2 and hence $l^2 x^2$ survives in the spectral sequence. Therefore there are no differentials in the Serre spectral sequence in this case. As before (using the same notations) we use the basis of $H^*(\mathbb{C}P^\infty)$ and $H^*(K(\mathbb{Z},3))$ to obtain the E_2-term and hence a basis for $H^*(E_{l^2})$: $\{1\otimes 1, 1\otimes x, i_3\otimes 1, 1\otimes x^2, i_3\otimes x, u\otimes 1, 1\otimes x^3, i_3^2\otimes 1, i_3\otimes x^2, u\otimes x, 1\otimes x^4, i_3^2\otimes x, i_3 u\otimes 1\}$ in bi-degrees $\{(0,0), (2,0), (0,3), (4,0), (2,3), (0,5), (6,0), (0,6), (4,3), (2,5), (8,0), (2,6), (0,8)\}$. Examining the different Steenrod operations in each dimension, we find that for the class in dimension 3 the parity of l plays a role. For dimensional reasons there are two possible operations: $s_q^1(i_3)$ and $s_q^2(i_3)$. Recall the definition of the Bockstein map, $\beta : H^i(X; \mathbb{Z}_2) \longrightarrow H^{i+1}(X; \mathbb{Z}_2)$, where $s_q^1(y) = \beta(y) = [\frac{\delta c}{2}]$. Here $y = [c]$ is a cocycle in $\mathbb{Z}_2$ represented by an integral class, also called $[c]$. We now claim that $s_q^1(i_3) = 0$. If not, then $s_q^1(i_3) = \beta(i_3) = x^2$ and therefore $2\,x^2 = 0$ in $H^4(E_{l^2})$. Note that $2\,x^2 \neq 0$ in $H^4(\mathbb{C}P^\infty)$ but it could still be zero in $H^4(E_{l^2})$. In the integral Serre spectral sequence for E_{l^2} the differential from i_3 to x^2 is multiplication by l^2. But l^2 is divisible by 4, hence $2\,x^2 \neq 0$ and therefore $s_q^1(i_3) = 0$. The remaining Steenrod operations are calculated with the help of their properties and we obtain that $H^*(E_{l^2})$ splits into the following.

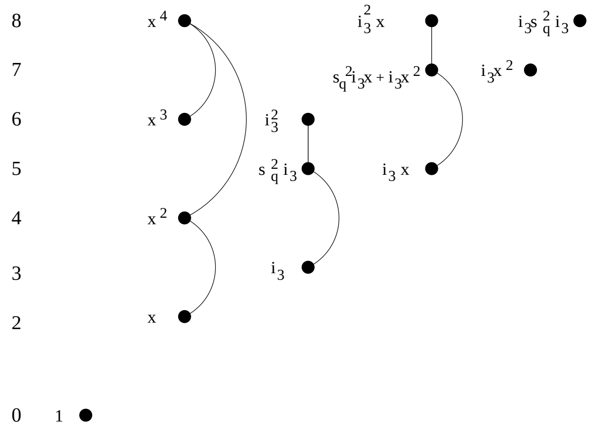

Figure 6. $H^*(E_{l^2})$ for l even

To compute $H^*(Th(\xi_k))$ we once again consider the Stiefel-Whitney classes. Modulo 2 we obtain $c(\xi_k) = 1 + x + x^4 + x^5 + \ldots$ for x a generator of $H^2(E_{l^2})$. As before we use the Thom isomorphism and properties of s_q^i to obtain the following summary of the Steenrod algebra structure of $H^*(Th(\xi_k))$.

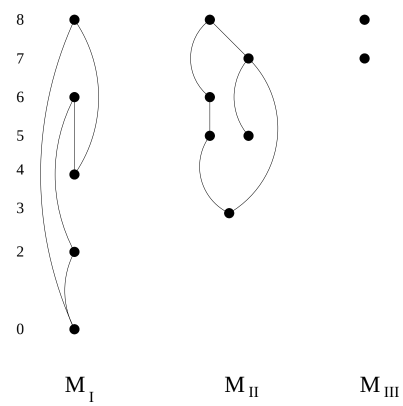

Figure 7. $H^*(Th(\xi_k))$ for l even

We now use R. Brunner's [B] computer program to compute $\mathrm{Ext}(M_I, \mathbb{Z}_2)$ and $\mathrm{Ext}(M_{II}, \mathbb{Z}_2)$.

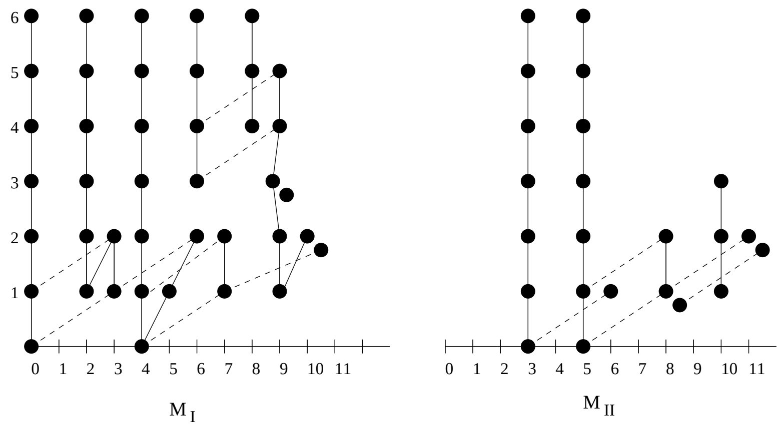

Figure 8. $\mathrm{Ext}^{s,t}(M_1;\mathbb{Z}_2)$ and $\mathrm{Ext}^{s,t}(M_2;\mathbb{Z}_2)$

First we note that there are no differentials between the long lines coming from the classes in dimensions seven and eight and between the long line in dimensions seven and the line in dimension six above $(6,2)$. The reason is that multiplication by h_0^2 is zero. Hence if a differential d_2 were non-zero then d_4 would be zero as d_4 can be derived from d_2 by multiplication with h_0^2: $d_4[(8,0)] = (7,4) = h_0^2\,(7,2) = h_0^2\, d_2[(8,0)]$. The remaining possible differentials are zero by h_0- and h_2-linearity. In order to explain exactly how many classes remain we use the higher order Bockstein differentials β_r. Assume that 2^i exactly divides l. Recall that $\beta_r : H^i(X;\mathbb{Z}_2) \longrightarrow H^{i+1}(X;\mathbb{Z}_2)$. We first take a cocycle $[c]$ in $\mathbb{Z}_2$ and represent it by an integral class, also call it $[c]$. We then take its coboundary, i.e. we find an integral class $[\gamma]$ of order l^2 such that $[\delta\, c] = [l^2\,\gamma]$. Note that this is possible as $H^4(E_{l^2};\mathbb{Z}) \cong \mathbb{Z}_{l^2}$, $H^3(E_{l^2};\mathbb{Z}) \cong 0$ and $H^4(E_{l^2};\mathbb{Z}_2) \cong \mathbb{Z}_2$, $H^3(E_{l^2};\mathbb{Z}_2) \cong \mathbb{Z}_2$. By definition we now obtain that $\beta_{2i}([c]) = [\frac{\delta\, c}{2^{2i}}] = [\frac{l^2}{2^{2i}}\,\gamma] \equiv \gamma \mod 2$ as $2^{2i}||l^2$. We now translate these Bockstein differentials on the cohomology level to the Ext level, i.e. the homotopy level. The goal is to first look at π_0 and then to look at a complex C_k with the property that

$$H_i(C_k) \cong \begin{cases} \mathbb{Z}_{2^k} & \text{if } i = 0; \\ 0 & \text{otherwise.} \end{cases}$$

This induces the exact sequence

$$S^0 \xrightarrow{\times 2^k} S^0 \longrightarrow C_k \longrightarrow S^1\,.$$

By the Hurewicz isomorphism we have that $\pi_0(C_k) \cong \mathbb{Z}_{2^k}$ for $k > 1$. We also get for the cohomology

$$H^i(C_k) \cong \begin{cases} \mathbb{Z}_{2^k} & \text{if } i = 0, 1; \\ 0 & \text{otherwise.} \end{cases}$$

We will use the following diagram of exact sequences.

$$\begin{tikzcd} S^0 \arrow[r, "\times 2^k"] \arrow[d, "\times 2^{k-1}"'] & S^0 \arrow[r] \arrow[d, "="'] & C_k \arrow[r] \arrow[d] & S^1 \arrow[d, "\times 2^k"'] \\ S^0 \arrow[r, "\times 2"] & S^0 \arrow[r] & C_1 \arrow[r] & S^1 \end{tikzcd}$$

On the cohomology level with coefficients in $\mathbb{Z}_2$ we obtain

$$\begin{tikzcd} H^*(S^0) \arrow[d, "="'] & H^*(C_1) \arrow[l] \arrow[d] & H^*(S^1) \arrow[l] \arrow[d, "0"'] \\ H^*(S^0) & H^*(C_k) \arrow[l] & H^*(S^1) \arrow[l] \end{tikzcd}$$

Chasing the diagram we obtain that $s_q^1 x_0 = 0$ in $H^*(C_k)$. Hence there are no Steenrod operations on $H^*(C_k)$ and we obtain for Ext just two shifted copies of $\mathrm{Ext}_{\mathcal{A}_2}(\mathbb{Z}_2, \mathbb{Z}_2)$, one for the class in dimension zero and one for the class in dimension one. As $\pi_0(C_k) \cong \mathbb{Z}_{2^k}$ we know that there must be non-zero differentials above k which come from the Bockstein differentials. For our case we are interested in $\Sigma^7 C_k$ where $k = 2\,i$ as our Bockstein multiplication is $2^k = 2^{2i}$. There is a map between Ext-groups which lets us use the above for $\Sigma^7 C_k$. Hence we obtain Bockstein differentials starting above $k = 2\,i$ between the long line in dimension eight coming from M_I and the long line in dimension seven coming from the class in dimension seven. Hence in computing $\pi_7^S(Th(\xi_k))$ we are left with $4+2i$ copies of $\mathbb{Z}_2$.

Conclusion 3. *$\pi_7^S(Th(\xi_k), \mathbb{Z}_2) \cong \mathbb{Z}_{2^{4+2i}} \oplus \mathbb{Z}_4$ where $2^i || l$.*

This completes the calculations for $\pi_7^S(Th(\xi_k, \mathbb{Z}_2))$.

6.3. **$\mathbf{p = 3}$ and 3 does not divide l.** For the higher primes we must repeat the above calculations. We first find $H^*(E_{l^2}; \mathbb{Z}_3)$ as a module over the Steenrod algebra. As before we use the fibration $K(\mathbb{Z}, 3) \longrightarrow E_{l^2} \longrightarrow \mathbb{C}P^\infty$. Then the E_2-term of the Serre spectral sequence for $H^*(E_{l^2}; \mathbb{Z}_3)$ is given by $E_2^{s,t} = H^s(\mathbb{C}P^\infty; \mathbb{Z}_3) \otimes H^t(K(\mathbb{Z}, 3); \mathbb{Z}_3)$. Recall that $H^*(K(\mathbb{Z}, 3); \mathbb{Z}_3)$ is a free, graded, commutative algebra on free generators $s_q^I\, i_n$ where i_n is in $H^n(K(\mathbb{Z}, 3); \mathbb{Z}_3)$ and I is an admissible sequence. Note that $H^s(\mathbb{C}P^\infty; \mathbb{Z}_3) \cong \mathbb{Z}_3[x]$ and $H^t(K(\mathbb{Z}, 3); \mathbb{Z}_3) \cong \mathbb{Z}_3[i_3, P^1\, i_3, \beta\, P^1\, i_3]/(i_3^2, (P^1 i_3^2)^2)$ where $P^i : H^k(X; \mathbb{Z}_3) \longrightarrow H^{k+4i}(X; \mathbb{Z}_3)$ and in our range the only possible operations are P^1 and P^2. Also recall that $\beta : H^i(X; \mathbb{Z}_3) \longrightarrow H^{i+1}(X; \mathbb{Z}_3)$ is the Bockstein homomorphism. We also have the following relations

$$(P^1)^2 = (\mathrm{unit}) \cdot P^2$$
$$\beta\,(x \cdot y) = \beta(x) \cdot y + x \cdot \beta(y)$$

$$P^i(x\cdot y)=\sum_{j+k=i}P^jx\cdot P^ky$$

as well as an instability axiom:

$$P^ix=\begin{cases}x^3 & \text{if } |x|=2i\\ 0 & \text{if } |x|<2i\,.\end{cases}$$

Using the Serre spectral sequence we obtain for the Steenrod algebra structure of $H^*(E_{l^2};\mathbb{Z}_3)$:

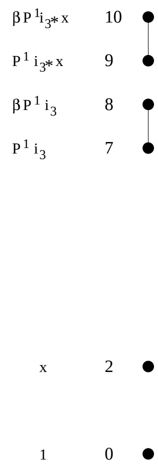

Figure 9. $H^*(E_{l^2};\mathbb{Z}_3)$ and $\gcd(3,l)=1$

As $c(\xi_k)=1$ in this case we get the same picture for $H^*(Th(\xi_k);\mathbb{Z}_3)$. Instead of calculating the Ext-groups we break up the calculation of the seventh stable homotopy group as follows.

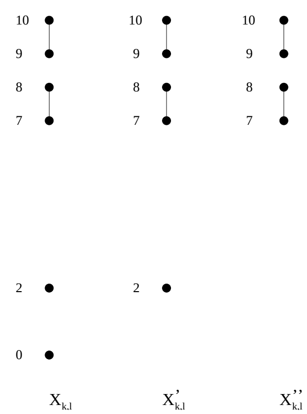

Figure 10. $H^*(Th(\xi_k);\mathbb{Z}_3)$ and $\gcd(3,l)=1$

Hence we get the following exact sequences.

$$S^0 \longrightarrow X_{k,l} \longrightarrow X'_{k,l} \tag{13}$$

$$S^2 \longrightarrow X'_{k,l} \longrightarrow X''_{k,l} \tag{14}$$

Let $X^{7,8}$ be the space with one generator in dimension 7 and one generator in dimension 8 and a s_q^1 between the two elements. As we are working with $\mathbb{Z}_3$ coefficients we obtain the following exact sequence.

$$S^7 \xrightarrow{\times 3} S^7 \longrightarrow X^{7,8} \longrightarrow S^8 \xrightarrow{\times 3} S^8 .$$

This translates to a homology exact sequence:

$$H_7(S^7) \xrightarrow{\times 3} H_7(S^7) \longrightarrow H_7(X^{7,8}) \longrightarrow 0 .$$

Hence $H_7(X^{7,8}) \cong \mathbb{Z}_3$ and by the Hurewicz isomorphism $\pi_7(X^{7,8}) \cong H_7(X^{7,8}) \cong \mathbb{Z}_3$ as $H_i(X^{7,8}) = 0$ for $i < 7$. The elements in dimensions 9 and 10 do not contribute to π_7, hence $\pi_7(X''_{k,l}) \cong \mathbb{Z}_3$. Using the above two sequences 13 and 14 to obtain associated exact sequences in homotopy we conclude that

$$\pi_7(Th(\xi_k); \mathbb{Z}_3) \cong \pi_7(X_{k,l}; \mathbb{Z}_3) \cong \begin{cases} \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \text{or} \\ \mathbb{Z}_9 , & \end{cases}$$

depending on an extension problem.

6.4. **p = 3 and 3 does divide l ($3^i||l$).** In this case there are no differentials in the Serre spectral sequence, i.e. all cycles are permanent and obtain the following for $H^*(E_{l^2}; \mathbb{Z}_3)$:

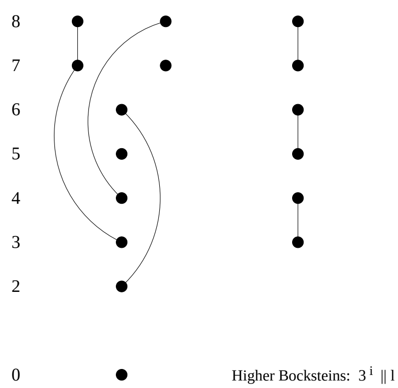

Figure 11. $H^*(E_{l^2}; \mathbb{Z}_3)$ and $3^i||l$

Here the Thom class translates under the Thom isomorphism into the Pontrjagin classes of ξ_k. But modulo 3 the total Pontrjagin class becomes 1 which means we obtain the same figure (Figure 11) for $H^*(Th(\xi_k); \mathbb{Z}_3)$. Breaking this diagram into its six different components for the calculation of Ext we obtain that

$$\pi_7(Th(\xi_k); \mathbb{Z}_3) \cong \mathbb{Z}_{3^{1+2i}} \oplus \mathbb{Z}_3 \quad \text{if } 3^i||l .$$

Remark 8. Note that higher Bockstein differentials exist between the long lines in the Ext-diagram coming from the classes in dimension eight and seven. The reason for their existence is the same as in the case of $p = 2$: for $[\gamma] \in H^4(E_{l^2}; \mathbb{Z}_3) \cong \mathbb{Z}_3$ and $[c] \in H^3(E_{l^2}; \mathbb{Z}_3) \cong \mathbb{Z}_3$ and $\delta\, c = l^2\, \gamma$ we obtain $\beta_{2i}([c]) = [\frac{\delta\, c}{3^{2i}}] = [\frac{l^2}{3^{2i}}\, \gamma] \equiv \gamma$ mod 3 as $3^{2i}||l^2$.

6.5. **p = 5 and 5 does not divide l.** Again we return to the Serre spectral sequence using the fact that $H^s(\mathbb{C}P^\infty; \mathbb{Z}_5) \cong \mathbb{Z}_5[x]$ and

$$H^t(K(\mathbb{Z},3); \mathbb{Z}_5) \cong \mathbb{Z}_5[i_3, P^1\, i_3, \beta\, P^1\, i_3]/(i_3^2, (P^1 i_3^2)^2)$$

where $P^i : H^k(X; \mathbb{Z}_5) \longrightarrow H^{k+8i}(X; \mathbb{Z}_5)$ and in our range the only possible operation is $P^1 : H^k(X; \mathbb{Z}_5) \longrightarrow H^{k+8}(X; \mathbb{Z}_5)$. In this case there are differentials in the Serre spectral sequence and the Steenrod algebra structure of $H^*(E_{l^2}; \mathbb{Z}_5)$ consists of one cycle in dimension zero and one in dimension two with no operations. As there are no P^1-operations, the structure for $H^*(Th(\xi_k); \mathbb{Z}_5)$ will be the same. Therefore we obtain

$$\pi_7(Th(\xi_k); \mathbb{Z}_5) \cong \pi_7(S^0 \vee S^2) \cong \mathbb{Z}_5\,.$$

6.6. **p = 5 and 5 does divide l ($5^i||l$).** In this case there are no differentials in the Serre spectral sequence. Hence all cycles are permanent and we obtain for $H^*(E_{l^2}; \mathbb{Z}_5)$ and therefore for $H^*(Th(\xi_k); \mathbb{Z}_5)$:

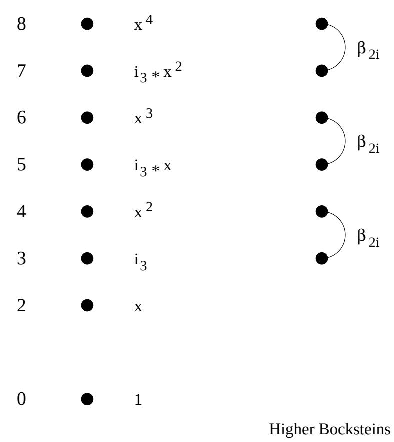

Figure 12. $H^*(E_{l^2}; \mathbb{Z}_5)$ and $5^i||l$

As these are all permanent cycles, the calculation for the Ext-groups breaks into eight pieces and we obtain that

$$\pi_7(Th(\xi_k); \mathbb{Z}_5) \cong \mathbb{Z}_{5^{2i}} \oplus \mathbb{Z}_5 \quad \text{if } 5^i||l\,.$$

6.7. **Higher primes.**

Claim 3. *For all primes $p \geq 7$*

$$\pi_7(Th(\xi_k);\mathbb{Z}_p) \cong \begin{cases} 0 & \text{if } gcd(p,l) = 1\,, \\ \mathbb{Z}_{p^{2i}} & \text{if } p^i || l\,. \end{cases}$$

Proof. We first consider the case of $\gcd(p, l) = 1$, $p \geq 7$. In this case, as in the case for $p = 5$, there are differentials in the Serre spectral sequence and the Steenrod algebra structure of $H^*(E_{l^2};\mathbb{Z}_p)$ and hence of $H^*(Th(\xi_k);\mathbb{Z}_p)$ consists of one cycle in dimension zero and one in dimension two with no operations. Hence

$$\pi_7(Th(\xi_k);\mathbb{Z}_p) \cong \pi_7(S^0 \vee S^2;\mathbb{Z}_p) \cong 0\,.$$

In the case of $p^i||l$ with $p \geq 7$ the picture for the Ext-groups looks just like the one for $p = 5$ but shifted by at least 2. In that case the single entry in dimension 7 will also shift by 2 and we obtain $\pi_7(Th(\xi_k);\mathbb{Z}_p) \cong \mathbb{Z}_{p^{2i}}$. $\square$

This completes the calculation for $\pi_7(Th(\xi_k))$ in the case of $l_1 = l_2 = l$. For the general case recall that

$$\begin{aligned} c(-\xi_k) &= [(1 + k\,u)^3\,(1 + e\,u)]^{-1} \\ &= 1 - (e + 3\,k)\,u + (e^2 + 3\,k\,e + 6\,k^2)\,u^2 \\ &\quad - (e^3 + 3\,k\,e^2 + 6\,k^2\,e + 10\,k^3)\,u^3 \\ &\quad + (e^4 + 3\,k\,e^3 + 6\,k^2\,e^2 + 10\,k^3\,e + 15\,k^4)\,u^4 - \cdots \end{aligned}$$

Here $e = l_1 + l_2$ and u is identified with its image in $H^2(E_{l_1 l_2})$. We again argue case by case according to the prime p used and the parity of k and l.

6.7.1. $p = 2$. First note that in the case of both l_1 and l_2 being odd, the calculation for $H^*(E_{l_1 l_2})$ as a module over the Steenrod algebra is exactly the same as in the above case 6.1 since this calculation only depended on the parity of $l_1\,l_2$. The corresponding calculation for both l_1 and l_2 being even also stays the same as it only depended on the divisibility of $l_1\,l_2$ by 4, see case 6.2.

If k is even and l_1 and l_2 are odd, then $e = l_1 + l_2$ is even. Hence $c(-\xi_k) = 1$ modulo 2 and therefore $w_i(-\xi_k) = 0$ for all $i > 0$. This implies that we get the same calculation for $H^*(Th(\xi_k))$as in the above case 6.1.1 of k even and l odd.

If k is odd and l_1 and l_2 are odd, then $e = l_1+l_2$ is even and $c(-\xi_k) = 1+u+u^4+\cdots$ which implies that $w_2(-\xi_k) = u, w_4(-\xi_k) = 0, w_6(-\xi_k) = 0, w_8(-\xi_k) = u^4$. Hence the calculation of $H^*(Th(\xi_k))$ is the same as in the above case 6.1.2 of k and l odd.

If k is odd and l_1 and l_2 are both even, then again $c(-\xi_k) = 1 + u + u^4 + \cdots$ which coincides with the above case 6.2 of k odd and l even.

However, if k is odd and exactly one of the l_i is odd, the calculation for $H^*(E_{l_1 l_2})$ as a module over the Steenrod algebra changes since $l_1\,l_2$ is no longer divisible by 4. Also, the calculation for $H^*(Th(\xi_k))$ changes accordingly. We defer this case to a sequel of this article.

6.7.2. $p = 3$. Recall that the calculation for $H^*(E_{l^2})$ as a module over the Steenrod algebra only depended on the divisibility of l^2 by 3. Hence the corresponding calculation for $H^*(E_{l_1 l_2})$ stays the same in all subcases. Recall that for the higher primes we need to compute the Pontrjagin classes of ξ_k. In the cases of $\gcd(3, l_1) = 1$, $\gcd(3, l_2) = 1$ and $3 \,|\, l_1$, $3 \,|\, l_1$ all classes vanish which coincides with the corresponding cases 6.3 and 6.4.

However, in the case of $(3, l_1) = 1$ and exactly one of the l_i is divisible by 3, the first and the second Pontrjagin class are non-trivial and we obtain a different calculation for $H^*(Th(\xi_k))$. We again defer this case.

6.7.3. $p = 5$ *and higher primes.* As in the case of $p = 3$ the corresponding calculation for $H^*(E_{l_1 l_2})$ stays the same in all subcases. As for dimensional reasons there are no operations, the calculations for $H^*(Th(\xi_k))$ stay the same.

7. Appendix 2: Analyzing the kernel of the Hurewicz map

We mainly use the following general fact about the image of the Hurewicz map.

Fact 1. [Sw, Chapter 19]*Let* $h_k^2 : \pi_k X \longrightarrow H_k(X;\mathbb{Z}) \longrightarrow H_k(X;\mathbb{Z}_2)$ *be the composition of the Hurewicz and the coefficient map. Then*

$$dim\{Im(h_k^2)\} \leq dim\{Ext^{0,k}(H^*(X,\mathbb{Z}_2);\mathbb{Z}_2)\}$$

where $Ext^{0,k}(H^*(X,\mathbb{Z}_2);\mathbb{Z}_2) = E_2^{0,k}$ *of the Adams spectral sequence for the* k*-th stable homotopy group* $\pi_k^S X$*. Equality holds if there are no differentials in Ext.*

Using this fact we return to the various cases, see Appendix 1.

7.1. **p = 2, l odd, k odd.** Let $X = Th(\xi_k)$. Note that $E_2^{0,7} = 0$ in this case and there are no differentials from $E_2^{0,7}$. Hence $\dim\{\mathrm{Im}(h_7^2)\} = 0$. But this implies that $\dim\{\mathrm{Im}(\pi_7 X \longrightarrow H_7(X;\mathbb{Z})\}$ is divisible by two. Recall that $H_7(E_{l^2};\mathbb{Z}) \cong \mathbb{Z}_{l^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong_{\text{Thom}} H_7(Th(\xi_k),\mathbb{Z})$. But this implies that the image of $\pi_7 X \longrightarrow H_7(X;\mathbb{Z})$ in the 2-torsion case lies in $\mathbb{Z}_2$ in $H_7(X;\mathbb{Z})$. But then this map must be trivial as it is divisible by 2. Hence the Hurewicz map is trivial modulo 2 and hence the kernel is everything:

$$\mathrm{Ker}(h) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{16}\,.$$

7.2. **p = 2, l odd, k even.** Here $E_2^{0,7} \neq 0$ and hence the Hurewicz map maps onto $\mathbb{Z}_2$. Then $\mathbb{Z}_2$ is not in the kernel of h and we obtain:

$$\mathrm{Ker}(h) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{16}$$

7.3. **$\mathbf{p=2}$, l even.** In this case $E_2^{0,7} \neq 0$ as $d_2[(7,0)] = 0$. We show that the Hurewicz map h is non-zero on $\mathbb{Z}_{2i}$. Recall that rational homology and rational stable homotopy of a spectrum X are the same, i.e. $H_*(X) \otimes \mathbb{Q} \cong \pi_*^S(X) \otimes \mathbb{Q}$. Integrally we have $\mathbb{Z} \subset \pi_k^S(X) \xrightarrow{h} H_k(X)$. If $x \in \pi_k^S(X)$ and if x is represented by an element of Ext in bidegrees $(s,t) = (0,k)$, then $h(x) \neq 0$ in $H_k(X;\mathbb{Z}_2)$. Recall that $\mathrm{Ext}_R^{0,k}(M,N) \cong \mathrm{Hom}_R^k(M,N) \cong \mathrm{Hom}(H_k(X;\mathbb{Z}_2), H_k(X;\mathbb{Z}_2))$. If $h(x)$ is a generator of $H_7(X)$, then the order of $h(x)$ divides the order of x. Since in our case $H_7(E_{l^2}) \cong \mathbb{Z}_{l^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ and $2^{2i}||l^2$, this generator x must have at least order 2^{2i} which implies that $\mathbb{Z}_{2i}$ must be in the image of h, hence it cannot be in the kernel. Hence we obtain:

$$\mathrm{Ker}(h) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{16}$$

7.4. **$\mathbf{p=3}$ and 3 does not divide l.** Using the fact that the Hurewicz map is surjective and that $H_7(E_{l^2};\mathbb{Z}_3) \cong \mathbb{Z}_3$ we conclude that $\mathrm{Ker}(h) \otimes \mathbb{Z}_3 \cong \mathbb{Z}_3$.

7.5. **$\mathbf{p=3}$ and 3 does divide l ($3^i||l$).** Following the same argument as in case 3 we conclude that $\mathbb{Z}_{3^{2i}}$ must be in the image of h and hence $\mathrm{Ker}(h) \otimes \mathbb{Z}_3 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

7.6. **$\mathbf{p=5}$ and 5 does not divide l.** As there is no $\mathbb{Z}_5$-factor in $H_7(E_{l^2})$, we obtain that $\mathrm{Ker}(h) \otimes \mathbb{Z}_5 \cong \mathbb{Z}_5$.

7.7. **$\mathbf{p=5}$ and 5 does divide l ($5^i||l$).** As $\mathbb{Z}_{5^{2i}}$ is contained in $H_7(E_{l^2})$ we conclude that $\mathrm{Ker}(h) \otimes \mathbb{Z}_5 \cong \mathbb{Z}_5$.

7.8. **Higher primes.** In the non-trivial case of $p^i||l$ for $p \geq 7$ the Hurewicz map is an isomorphism, hence in both cases the kernel of h is trivial.

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