

# TOPOLOGY OF NON-NEGATIVELY CURVED MANIFOLDS

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An important question in the study of Riemannian manifolds of positive sectional curvature is how to distinguish manifolds that admit a metric with non-negative sectional curvature from those that admit one of positive curvature. Surprisingly, if the manifolds are compact and simply connected, all known obstructions to positive curvature are already obstructions to non-negative curvature. On the other hand, there are very few known examples of manifolds with positive curvature. They consist, apart from the rank one symmetric spaces, of certain homogeneous spaces  $G/H$  in dimensions 6, 7, 12, 13 and 24 due to Berger [Be], Wallach [Wa], and Aloff-Wallach [AW], and of biquotients  $K\backslash G/H$  in dimensions 6, 7 and 13 due to Eschenburg [E1],[E2] and Bazaikin [Ba], see [Zi] for a survey. Recently, a new example of a positively curved 7-manifold was found which is homeomorphic but not diffeomorphic to the unit tangent bundle of  $\mathbb{S}^4$ , see [GVZ, De]. And in [PW] a method was proposed to construct a metric of positive curvature on the Gromoll-Meyer exotic 7-sphere.

Among the known examples of positive curvature there are two infinite families: in dimension 7 one has the homogeneous Aloff-Wallach spaces, and more generally the Eschenburg biquotients, and in dimension 13 the Bazaikin spaces. The topology of these manifolds has been studied extensively, see [KS1, KS2, AMP1, AMP2, Kr1, Kr2, Kr3, Sh, CEZ, FZ]. There exist many 7-dimensional positively curved examples which are homeomorphic to each other but not diffeomorphic, whereas in dimension 13, they are conjectured to be diffeomorphically distinct [FZ].

In contrast to the positive curvature setting, there exist comparatively many examples with non-negative sectional curvature. The bi-invariant metric on a compact Lie group  $G$  induces, by O'Neill's formula, non-negative curvature on any homogeneous space  $G/H$  or more generally on any biquotient  $K\backslash G/H$ . In [GZ1] a large new family of cohomogeneity one manifolds with non-negative curvature was constructed, giving rise to non-negatively curved metrics on exotic spheres. Hence it is natural to ask whether, among the known examples, it is possible to topologically distinguish manifolds with non-negative curvature from those admitting positive curvature. The purpose of this article is to address this question. There are many examples of non-negatively curved manifolds which are not homotopy equivalent to any of the known positively curved examples simply because they have different cohomology rings. But recently new families of non-negatively curved manifolds were discovered [GZ2] which, as we will see, give rise to several new manifolds having the same cohomology ring as the 7-dimensional Eschenburg spaces.

Recall that the Eschenburg biquotients are defined as

$$E_{k,l} = \text{diag}(z^{k_1}, z^{k_2}, z^{k_3}) \backslash \text{SU}(3) / \text{diag}(z^{l_1}, z^{l_2}, z^{l_3})^{-1},$$

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with  $k := (k_1, k_2, k_3)$ ,  $l := (l_1, l_2, l_3)$ ,  $k_i, l_i \in \mathbb{Z}$ ,  $\sum k_i = \sum l_i$ , and  $z \in S^1 \subset \mathbb{C}$ . They include the homogeneous Aloff-Wallach spaces  $W_{a,b} = \text{SU}(3)/\text{diag}(z^a, z^b, \bar{z}^{a+b})$  and the manifolds  $F_{a,b}$  with  $k = (a, b, a+b)$  and  $l = (0, 0, 2(a+b))$ . Under certain conditions on  $k, l$ , see (1.2), they admit positive sectional curvature.

We will consider compact simply connected seven dimensional manifolds  $M$  whose non-trivial cohomology groups consist of  $H^i(M; \mathbb{Z}) \cong \mathbb{Z}$  for  $i = 0, 2, 5, 7$  and  $H^4(M, \mathbb{Z}) \cong \mathbb{Z}_r$ ,  $r \geq 1$ , where the square of a generator of  $H^2(M; \mathbb{Z})$  generates  $H^4(M, \mathbb{Z})$ . If in addition  $M$  is spin, we say that  $M$  has *cohomology type*  $E_r$  and if  $M$  is non-spin, *cohomology type*  $\bar{E}_r$ . In [E1] it was shown that an Eschenburg space is of cohomology type  $E_r$  with  $r = |\sigma_2(k) - \sigma_2(l)|$ , where  $\sigma_i(k)$  stands for the elementary symmetric polynomial of degree  $i$  in  $k_1, k_2, k_3$ . M. Kreck and S. Stolz defined certain invariants and showed that they classify manifolds of cohomology type  $E_r$  and  $\bar{E}_r$  up to homeomorphism and diffeomorphism [Kr, KS1, KS2]. The invariants were computed for most Eschenburg spaces in [Kr3, CEZ].

One class of manifolds we will study are the total spaces of 3-sphere bundles over  $\mathbb{C}\mathbb{P}^2$ . They fall into two categories, bundles which are not spin,

$$\mathbb{S}^3 \rightarrow S_{a,b} \rightarrow \mathbb{C}\mathbb{P}^2, \quad p_1 = 2a + 2b + 1, \quad e = a - b, \quad w_2 \neq 0$$

and bundles which are spin

$$\mathbb{S}^3 \rightarrow \bar{S}_{a,b} \rightarrow \mathbb{C}\mathbb{P}^2, \quad p_1 = 2a + 2b, \quad e = a - b, \quad w_2 = 0.$$

They are specified by the value of their Pontryagin class, Euler class and Stiefel-Whitney class. Here  $a, b$  are arbitrary integers.

A second class of manifolds can be described as follows. Consider the bundles

$$\mathbb{C}\mathbb{P}^1 \rightarrow N_t \rightarrow \mathbb{C}\mathbb{P}^2, \quad \mathbb{S}^1 \rightarrow M_{a,b}^t \rightarrow N_t.$$

Here  $N_t$  is the  $\mathbb{S}^2$  bundle with Pontryagin class  $p_1 = 1 - 4t$  and  $w_2 \neq 0$ , and  $M_{a,b}^t$  the circle bundle classified by the Euler class  $e = ax + by$  in terms of some natural basis  $x, y \in H^2(N_t, \mathbb{Z}) \cong \mathbb{Z}^2$ , where  $a, b$  are relatively prime integers.

Similarly, let

$$\mathbb{C}\mathbb{P}^1 \rightarrow \bar{N}_t \rightarrow \mathbb{C}\mathbb{P}^2, \quad \mathbb{S}^1 \rightarrow \bar{M}_{a,b}^t \rightarrow N_t$$

where  $\bar{N}_t$  is the  $\mathbb{S}^2$  bundle with Pontryagin class  $p_1 = 4t$ ,  $w_2 = 0$ , and  $\bar{M}_{a,b}^t$  the circle bundle with Euler class described by  $a, b$  as above. We will show:

**THEOREM.** *The above manifolds have the following properties:*

- (a)  $S_{a,b}$ ,  $M_{a,b}^t$  and  $\bar{M}_{a,2b}^t$  have cohomology type  $E_r$ , and  $\bar{S}_{a,b}$ ,  $\bar{M}_{a,2b+1}^t$  have cohomology type  $\bar{E}_r$ .
- (b)  $S_{a,b}$ ,  $M_{a,b}^t$  and  $\bar{M}_{a,b}^{2t}$  admit metrics with non-negative sectional curvature.
- (c)  $M_{a,1-a}^t$  is diffeomorphic to  $S_{-t,a(a-1)}$ , and  $\bar{M}_{a,1}^t$  is diffeomorphic to  $\bar{S}_{t,a^2}$ .
- (d)  $M_{a,b}^1$  is the Aloff-Wallach space  $W_{a,b}$  with base space  $N_1 = \text{SU}(3)/\mathbb{T}^2$ , and  $M_{a,b}^{-1} = F_{a,b}$  with base space the biquotient  $N_{-1} = \text{SU}(3)//\overline{\mathbb{T}^2}$ .
- (e)  $M_{a,b}^0$  is the set of circle bundles over  $N_0 = \mathbb{C}\mathbb{P}^3 \# \mathbb{C}\mathbb{P}^3$  and  $\bar{M}_{a,b}^0$  the set of circle bundles over  $\bar{N}_0 = \mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^1$ .

The existence of the metrics in part (b) will follow from [GZ2] after we describe the manifolds in a different fashion, namely as quotients of certain  $U(2)$  principal bundles over  $\mathbb{C}\mathbb{P}^2$ . Part (c) and (d) imply that the circle bundles  $M_{a,b}^1$ ,  $ab(a+b) \neq 0$ , and  $M_{a,b}^{-1}$ ,  $ab > 0$  as well as the sphere bundles  $S_{-1,a(a-1)}$ ,  $a \geq 2$  naturally admit a metric with positive sectional curvature. By computing the Kreck-Stolz invariants, we obtain a diffeomorphism classification of the above four classes of 7-manifolds and by comparing them to the invariants for Eschenburg spaces, we will obtain many other diffeomorphisms of  $M_{a,b}^t$  and  $S_{a,b}$  to positively curved Eschenburg spaces. For example:

- $S_{a,b}$  with  $a - b = 41$  is diffeomorphic to a positively curved Eschenburg space if and only if  $b \equiv 2285$  or  $5237 \pmod{6888}$ . In this case it is diffeomorphic to the cohomogeneity two Eschenburg space  $E_{k,l}$  with  $k = (2, 3, 7)$ ,  $l = (12, 0, 0)$ .
- $M_{a,b}^t$  with  $(a, b, t) = (638, -607, -403)$  is diffeomorphic to the positively curved cohomogeneity two Eschenburg space  $E_{k,l}$  with  $k = (1, 2, 5)$ ,  $l = (8, 0, 0)$ .

See Section 7 and Table A and B for further examples. We will also obtain a description of which Eschenburg spaces can be diffeomorphic to  $\mathbb{S}^3$  bundles over  $\mathbb{C}\mathbb{P}^2$ , see Theorem 7.1.

The examples of diffeomorphisms above imply that besides the metric of non-negative curvature the manifolds  $S_{a,b}$  and  $M_{a,b}^t$  sometimes admit a very different metric which has positive curvature. This raises the question whether perhaps they all admit a metric of positive curvature.

Some of these manifolds are also known to admit Einstein metrics. In [W] M.Wang showed that the Aloff-Wallach spaces  $M_{a,b}^1$ , and with W.Ziller in [WZ] that the circle bundles  $\bar{M}_{a,b}^0$  over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^2$ , all admit Einstein metrics. In [Che] D. Chen proved that the sphere bundles  $S_{a,b}$  and  $\bar{S}_{a,b}$  admit an Einstein metric if the structure group reduces from  $SO(4)$  to  $T^2 \subset SO(4)$ . Diffeomorphisms within each of these 3 classes of Einstein manifolds have been considered in [KS1, KS2, Che]. Using our computation of the invariants, we also find examples of diffeomorphism between different classes:

- The two Einstein manifolds  $\bar{M}_{70,5899}^0$  and  $\bar{S}_{62500,57600}$  are diffeomorphic to each other.
- For the sphere bundle  $\bar{S}_{a,b}$  with  $(a, b) = (q^2, 0)$  the structure group reduces to a 2-torus. This Einstein manifold is diffeomorphic to the Einstein manifold  $\bar{M}_{q,1}^0$ .

We will also see that among some of these classes there are no diffeomorphisms:

- There are no diffeomorphisms between the spin Einstein manifolds  $M_{a,b}^1$  and  $\bar{M}_{a',2b'}^0$ .
- There are no diffeomorphisms between the spin Einstein manifolds  $S_{a,b}$  and either  $M_{a',b'}^1$  or  $\bar{M}_{a',2b'}^0$  or an Eschenburg space.

Here is a short description of the content of the paper. In Section 1 we collect preliminaries and in Section 2 we recall the formulas for the Kreck-Stolz invariants. In Section 3 we describe the topology of the sphere bundles  $S_{a,b}$  and their invariants and in Section 4 the topology of the circle bundles  $M_{a,b}^t$ . In Section 5 we discuss the geometry of both families and in Section 6 we examine the manifolds  $\bar{S}_{a,b}$  and  $\bar{M}_{a,b}^t$ . In Section 7 we apply these results to obtain various examples of diffeomorphisms as described above.

## 1. PRELIMINARIES

We will compare several classes of manifolds of non-negative curvature with the family of positively curved Eschenburg spaces  $E_{k,l}$  described in the introduction. In order for an

Eschenburg space to be a manifold, i.e. in order for the  $S^1$  action on  $SU(3)$  to be free, we need

$$(1.1) \quad \gcd(k_1 - l_i, k_2 - l_j) = 1, \text{ for all } i \neq j, i, j \in \{1, 2, 3\}.$$

The *Eschenburg metric* on  $E_{k,l}$  is the submersion metric obtained by scaling the bi-invariant metric on  $SU(3)$  in the direction of a subgroup  $U(2) \subset SU(3)$  by a constant less than 1. It has positive sectional curvature [E2] if and only if, for all  $1 \leq i \leq 3$ ,

$$(1.2) \quad k_i \notin [\min(l_1, l_2, l_3), \max(l_1, l_2, l_3)], \text{ or } l_i \notin [\min(k_1, k_2, k_3), \max(k_1, k_2, k_3)].$$

If this condition is satisfied, we call  $E_{k,l}$  a *positively curved Eschenburg space*.

There are two subfamilies of Eschenburg spaces that are of interest to us. One is the family of homogeneous Aloff-Wallach spaces  $W_{p,q} = SU(3)/\text{diag}(z^p, z^q, \bar{z}^{p+q})$ , where  $p, q \in \mathbb{Z}$  with  $(p, q) = 1$ . This space has a homogeneous metric with positive sectional curvature if and only if  $pq(p+q) \neq 0$ . By interchanging coordinates, and replacing  $z$  by  $\bar{z}$  if necessary, we can assume that  $p \geq q \geq 0$ , and thus  $W_{1,0}$  is the only Aloff-Wallach space that does not admit a homogeneous metric with positive curvature. The second family consists of the Eschenburg biquotients  $F_{p,q} = E_{k,l}$  with  $k = (p, q, p+q)$  and  $l = (0, 0, 2p+2q)$  with  $(p, q) = 1$ . We can also assume that  $p \geq q$ , but here  $p$  and  $q$  can have opposite sign. The Eschenburg metric on  $F_{p,q}$  has positive sectional curvature if and only if  $pq > 0$ . These two families of Eschenburg spaces are special in that they admit circle fibrations over positively curved 6-manifolds. In the case of the Aloff-Wallach spaces, the base of the fibration is the homogeneous flag manifold  $SU(3)/T^2$  and in case of the Eschenburg biquotients  $F_{p,q}$ , the base is the inhomogeneous Eschenburg flag manifold  $SU(3)//T^2 := \text{diag}(z, w, zw) \setminus SU(3)/\text{diag}(1, 1, z^2w^2)^{-1}$ ,  $|z| = |w| = 1$ .

We also use the fact that the two Eschenburg spaces  $W_{1,1}$  and  $F_{1,1}$  can be regarded as principal  $SO(3)$  bundles over  $\mathbb{C}P^2$ , see [Sh, Cha].

Recall that we say that a compact simply connected seven dimensional spin manifold  $M$  has *cohomology type*  $E_r$ , or simply is of type  $E_r$ , if its cohomology ring is given by:

$$(1.3) \quad H^0(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z}) \cong H^5(M; \mathbb{Z}) \cong H^7(M; \mathbb{Z}) \cong \mathbb{Z} \text{ and } H^4(M; \mathbb{Z}) \cong \mathbb{Z}_r.$$

with  $r \geq 1$ . Furthermore, if  $u$  is a generator of  $H^2(M; \mathbb{Z})$ , then  $u^2$  is a generator of  $H^4(M; \mathbb{Z})$ , if  $r > 1$ . If a manifold with cohomology ring (1.3) is non-spin, we say it has type  $\bar{E}_r$ .

As was shown in [E1], an Eschenburg space is of type  $E_r$  with  $r = |\sigma_2(k) - \sigma_2(l)|$  where  $\sigma_i(k)$  stands for the elementary symmetric polynomial of degree  $i$  in  $k_1, k_2, k_3$ . Furthermore, for an Eschenburg space  $r$  is always odd (see Theorem 2.6), and examples of Eschenburg spaces exist for any odd number  $r \geq 3$ , for example the cohomogeneity one manifolds  $k = (p, 1, 1)$  and  $l = (p+2, 0, 0)$  with  $r = 2p+1$ .

Let  $M$  be a manifold of cohomology type  $E_r$  or  $\bar{E}_r$ . If we fix a generator  $u$  of  $H^2(M; \mathbb{Z})$ , the class  $u^2$  is a generator of  $H^4(M; \mathbb{Z})$ , which does not depend on the choice of  $u$ . We can thus identify the first Pontryagin class  $p_1(TM) \in H^4(M; \mathbb{Z}) \cong \mathbb{Z}_r$  with a well defined integer modulo  $r$ . Furthermore, it is a homeomorphism invariant. In the case of an Eschenburg space this integer is  $p_1(TE_{k,l}) = 2\sigma_1(k)^2 - 6\sigma_2(k) \pmod r$ , see [Kr2].

For manifolds of type  $E_r$  or  $\bar{E}_r$  we also have, besides  $r$ , a second homotopy invariant given by the linking form  $Lk$ . The linking form is a quadratic form on  $H^4(M; \mathbb{Z}) \cong \mathbb{Z}_r$ ,

with values in  $\mathbb{Q}/\mathbb{Z}$ . It is determined by the self-linking number  $lk(M) := Lk(u^2, u^2)$  of the generator  $u^2$  of  $H^4(M; \mathbb{Z})$ . Following [B]  $Lk(u^2, u^2)$  can be written in terms of the Bockstein homomorphism  $\beta : H^3(M; \mathbb{Q}/\mathbb{Z}) \rightarrow H^4(M; \mathbb{Z})$  which is associated to the exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$  :

$$Lk(u^2, u^2) := \langle \beta^{-1}(u^2), u^2 \cap [M] \rangle .$$

Since in our case  $H^4(M; \mathbb{Z}) \cong \mathbb{Z}_r$ , it follows that  $Lk(u^2, u^2)$  is a unit in the subgroup of order  $r$  in  $\mathbb{Q}/\mathbb{Z}$  and we can thus interpret  $lk(M)$  as an integer modulo  $r$ . Unlike  $p_1(TM)$ , the linking form  $lk(M)$  is orientation sensitive. In the case of an Eschenburg space we have  $lk(E_{k,l}) = \pm s^{-1} \bmod r$ , where  $s = \sigma_3(k) - \sigma_3(l)$  and  $s^{-1}$  is the multiplicative inverse in  $\mathbb{Z}_r$ , which is well defined since the freeness condition implies  $(s, r) = 1$ , see [Kr2].

As was observed by Kruggel, manifolds of type  $E_r$  or  $\bar{E}_r$  fall into two further homotopy types since  $\pi_4(M)$  can only be 0 or  $\mathbb{Z}_2$ . Indeed, if we consider the circle bundle  $S^1 \rightarrow G \rightarrow M$  whose Euler class is a generator of  $H^2(M, \mathbb{Z}) = \mathbb{Z}$ , it follows that  $H^*(G, \mathbb{Z}) \cong H^*(S^3 \times S^5, \mathbb{Z})$  and hence the attaching map of the 5 cell to the 3-skeleton is an element of  $\pi_4(S^3) \cong \mathbb{Z}_2$ . If the attaching map is trivial,  $G$  has the homotopy type of  $S^3 \times S^5$  and hence  $\pi_4(M) \cong \mathbb{Z}_2$ . A second manifold  $G$  with this homology is  $G = \text{SU}(3)$  and since  $\pi_4(\text{SU}(3)) = 0$ , its attaching map is non-trivial. Thus  $S^3 \times S^5$  and  $\text{SU}(3)$  are the only two possible homotopy types for  $G$ .

For Eschenburg spaces we have  $\pi_4(E_{k,l}) = 0$  since  $\pi_4(\text{SU}(3)) = 0$ .

Lastly, we discuss the relationship between principal  $\text{SO}(3)$  and  $\text{SO}(4)$  bundles and their classification if the base is  $\mathbb{C}\mathbb{P}^2$ , see [DW] and [GZ2]. Recall that  $\text{SO}(4) = S^3 \times S^3 / \{\pm(1, 1)\}$  defined by left and right multiplication of unit quaternions on  $\mathbb{H} \cong \mathbb{R}^4$ . Thus there are two normal subgroups

$$S_-^3 = S^3 \times \{e\}, S_+^3 = \{e\} \times S^3 \subset \text{SO}(4) = S^3 \times S^3 / \{\pm(1, 1)\}$$

isomorphic to  $S^3$  and  $\text{SO}(4)/S_\pm^3$  is isomorphic to  $\text{SO}(3)$ . Hence, if  $\text{SO}(4) \rightarrow P \rightarrow M$  is a principal  $\text{SO}(4)$  bundle, there are two associated principal  $\text{SO}(3)$ -bundles

$$\text{SO}(3) \rightarrow P_\pm := P/S_\pm^3 \rightarrow M \text{ with } \text{SO}(3) = S^3/\{\pm 1\}.$$

If  $M$  is compact and simply connected,  $P$  is uniquely determined by the  $\text{SO}(3)$  bundles  $P_\pm$ , see [GZ2], Proposition 1.8. For the characteristic classes one has

$$(1.4) \quad p_1(P_\pm) = p_1(P) \pm 2e(P), \quad w_2(P) = w_2(P_\pm).$$

One can see this on the level of classifying spaces by computing the maps induced in cohomology in the commutative diagram

$$\begin{array}{ccc} B_{S^3 \times S^3} & \longrightarrow & B_{\text{SO}(4)} \\ \downarrow \pi_\pm & & \downarrow /S_\pm^3 \\ B_{S^3} & \longrightarrow & B_{\text{SO}(3)} \end{array}$$

where  $\pi_\pm$  are induced by the projections onto the first and second factor. On the level of maximal tori one has  $\text{diag}(e^{i\theta}, e^{i\psi}) \subset S^3 \times S^3 \rightarrow \text{diag}(R(\theta - \psi), R(\theta + \psi)) \subset \text{SO}(4)$  where  $R(\theta)$  is a rotation by angle  $\theta$ . Thus in the natural basis of the second cohomology of the maximal tori,  $x, y$  in the case of  $\text{SO}(4)$  and  $r, s$  in the case of  $S^3 \times S^3$ , one has  $x \rightarrow r - s$  and  $y \rightarrow r + s$  and since  $p_1(P) = x^2 + y^2$  and  $e(P) = xy$ , we have  $p_1(P) + 2e(P) = 4r^2$  and

$p_1(P) - 2e(P) = 4s^2$ . The claim now follows by observing that the map  $H^4(B_{\text{SO}(3)}, \mathbb{Z}) \cong \mathbb{Z} \rightarrow H^4(B_{\text{S}^3}, \mathbb{Z}) \cong \mathbb{Z}$  is multiplication by 4. Notice that this corrects a mistake in [GZ2], (1.10), where  $P_{\pm}$  was defined as  $P/S_{\mp}^3$ . This will be crucial for us in Section 6.

It is also important for us to understand in detail the above discussion in the context of  $U(2)$  principal bundles. Assume that the structure group of an  $\text{SO}(4)$  principal bundle  $P$  reduces to  $U(2)$ :

$$U(2) \rightarrow P^* \rightarrow M \text{ and } P = P^* \times_{U(2)} \text{SO}(4),$$

or equivalently the 4-dimensional vector bundle corresponding to  $P$  has a complex structure. We identify  $\mathbb{C} \oplus \mathbb{C} \cong \mathbb{H}$  via  $(u, v) \rightarrow u + vj$  so that left multiplication by  $z$  is the usual complex structure on  $\mathbb{R}^4$ . This defines the embedding  $U(2) \subset \text{SO}(4)$  and implies that  $U(2) = S^1 \times S^3 / \{\pm(1, 1)\} \subset S^3 \times S^3 / \{\pm(1, 1)\} = \text{SO}(4)$ . Notice also that the image of  $S^1 \times \{e\}$  is the center of  $U(2)$ , and the image of  $\{e\} \times S^3$  is  $\text{SU}(2) \subset U(2)$ .

For  $P^*$  we have the Chern classes  $c_1$  and  $c_2$  and for the underlying real bundle one has  $p_1(P) = c_1^2 - 2c_2$ ,  $e(P) = c_2$ , and  $w_2(P) = c_1 \pmod{2}$ . Thus (1.4) implies that

$$(1.5) \quad p_1(P_-) = c_1^2 - 4c_2 \quad , \quad p_1(P_+) = c_1^2 \quad \text{and} \quad w_2(P_{\pm}) \equiv c_1 \pmod{2}.$$

We now discuss the relationship between the associated  $\text{SO}(3)$  principal bundles  $P_{\pm}$  and the  $U(2)$ -reduction  $P^*$  and claim that:

$$(1.6) \quad P_- = P^*/Z \quad , \quad P_+ = (P^*/\text{SU}(2)) \times_{\text{SO}(2)} \text{SO}(3)$$

where  $Z$  is the center of  $U(2)$ . Indeed, if we set  $\Gamma = \{\pm(1, 1)\}$  when in  $S^3 \times S^3$  and  $\Gamma = \{\pm 1\}$  when in  $S^3$ , we get

$$\begin{aligned} P_- = P/S_-^3 &= [P^* \times_{U(2)} \text{SO}(4)] / S^3 \times \{e\} = [P^* \times_{(S^1 \times S^3)/\Gamma} (S^3 \times S^3)/\Gamma] / S^3 \times \{e\} \\ &= P^* \times_{(S^1 \times S^3)/\Gamma} [\{e\} \times (S^3/\Gamma)] = P^* / [(S^1/\Gamma) \times \{e\}] = P^*/Z \end{aligned}$$

and for the second bundle

$$\begin{aligned} P_+ = P/S_+^3 &= [P^* \times_{(S^1 \times S^3)/\Gamma} (S^3 \times S^3)/\Gamma] / \{e\} \times S^3 = P^* \times_{(S^1 \times S^3)/\Gamma} [(S^3/\Gamma) \times \{e\}] \\ &= (P^*/\text{SU}(2)) \times_{(S^1/\Gamma) \times \{e\}} [(S^3/\Gamma) \times \{e\}] = (P^*/\text{SU}(2)) \times_{\text{SO}(2)} \text{SO}(3) \end{aligned}$$

Thus the structure group of  $P_+$  reduces to  $\text{SO}(2)$ . The principal bundle of this reduced bundle is

$$(1.7) \quad S^1 \cong U(2)/\text{SU}(2) \rightarrow P^*/\text{SU}(2) \rightarrow P^*/U(2) \quad \text{with Euler class } e = c_1(P^*)$$

To see that this bundle indeed has Euler class  $c_1(P^*)$ , we consider the commutative diagram of classifying spaces:

$$\begin{array}{ccc} P^* & \longrightarrow & B_{U(2)} \\ \downarrow & & \downarrow \text{det} \\ P^*/SU(2) & \longrightarrow & B_{S^1} \end{array}$$

The isomorphism  $U(2)/SU(2) \cong S^1$  is induced by the homomorphism  $\text{det}: U(2) \rightarrow S^1$ . Since  $\text{det}(\text{diag}(e^{i\theta}, e^{i\psi})) = e^{i(\theta+\psi)}$ , the induced map  $B_{U(2)} \rightarrow B_{S^1}$  in cohomology takes  $z \rightarrow x + y$  in the natural basis of the cohomology of the maximal tori, and since  $c_1 = x + y$ , the claim follows.

We now specialize to the case where the base is 4-dimensional. Principal  $SO(4)$  bundles  $P$  over a compact simply connected 4-manifold are classified by the characteristic classes  $p_1(P)$ ,  $e(P)$  and  $w_2(P)$ , and principal  $SO(3)$  bundles by  $p_1(P)$  and  $w_2(P)$ , see [DW]. But these classes cannot be assigned arbitrarily. To describe the restriction, we identify  $p_1(P)$  and  $e(P)$  with an integer, using a choice of an orientation class. For an  $SO(3)$  principal bundle the value of  $w_2(P)$  is arbitrary. The value of  $p_1(P)$  on the other hand satisfies  $p_1(P) \equiv e^2 \pmod{4}$  where  $e \in H^2(M, \mathbb{Z})$  is the Euler class of a principal circle bundle with  $e \equiv w_2(P) \pmod{2}$ . Via equation (1.4) this completely describes the possible values of the invariants for principal  $SO(4)$  bundles as well.

In the case of bundles over  $\mathbb{C}\mathbb{P}^2$ , one thus has the following. Throughout the article, we use the generator  $x$  of  $H^2(\mathbb{C}\mathbb{P}^2, \mathbb{Z})$  which is given by the Euler class of the Hopf bundle. The cohomology class  $x^2$  is our choice of an orientation class in  $H^4(\mathbb{C}\mathbb{P}^2, \mathbb{Z})$ . The invariants  $p_1$  and  $e$  are then identified with integers by evaluation on the fundamental class. For principal  $SO(3)$  bundles one then has

$$(1.8) \quad p_1(P) \equiv 1 \pmod{4} \quad \text{if } w_2(P) \neq 0 \quad \text{and} \quad p_1(P) \equiv 0 \pmod{4} \quad \text{if } w_2(P) = 0.$$

For an  $SO(4)$  principal bundle  $P$  with  $w_2(P) = w_2(P_{\pm}) \neq 0$  one thus has  $p_1(P_-) = 4a + 1$  and  $p_1(P_+) = 4b + 1$  for some  $a, b \in \mathbb{Z}$  and hence

$$(1.9) \quad p_1(P) = 2a + 2b + 1 \quad , \quad e(P) = a - b \quad \text{if } w_2(P) \neq 0$$

If on the other hand  $w_2(P) = 0$ , one has  $p_1(P_-) = 4a$  and  $p_1(P_+) = 4b$  for some  $a, b \in \mathbb{Z}$  and hence

$$(1.10) \quad p_1(P) = 2a + 2b \quad , \quad e(P) = a - b \quad \text{if } w_2(P) = 0.$$

We describe the bundles by specifying these two (arbitrary) integers  $a, b$ .

In the case of  $U(2)$  bundles over  $\mathbb{C}\mathbb{P}^2$ , they are classified by  $c_1 = rx$  and  $c_2 = sx^2$  and  $r, s \in \mathbb{Z}$  can be chosen arbitrarily. The structure group of an  $SO(4)$  bundle reduces to  $U(2)$  if and only if  $p_1(P_+)$  is a square, and to  $T^2 \subset SO(4)$  if and only if both  $p_1(P_+)$  and  $p_1(P_-)$  are squares.

We will also use the fact that for an  $SO(3)$  principal bundle  $P$  over  $\mathbb{C}\mathbb{P}^2$  one has

$$(1.11) \quad |p_1(P)| = |H^4(P, \mathbb{Z})| \quad \text{and} \quad \pi_1(P) = 0 \quad \text{if } w_2 \neq 0 \quad \text{or} \quad \pi_1(P) = \mathbb{Z}_2 \quad \text{if } w_2 = 0$$

see [GZ2, Proposition 3.6].

## 2. KRECK-STOLZ INVARIANTS

The Kreck-Stolz invariants are based on the Eells-Kuiper  $\mu$ -invariant and are defined as linear combinations of relative characteristic numbers of appropriate bounding manifolds. They were introduced and calculated for certain homogeneous spaces  $L_{a,b}$  in [KS1], for the Aloff-Wallach spaces in [KS2], and for most of the Eschenburg spaces in [Kr2]. In the case of smooth simply connected closed seven dimensional manifolds  $M$  of cohomology type  $E_r$  or  $\bar{E}_r$ , the Kreck-Stolz invariants provide a classification up to homeomorphism and diffeomorphism.

Let  $M$  be a seven dimensional oriented manifold of cohomology type  $E_r$  or  $\bar{E}_r$  and  $u \in H^2(M, \mathbb{Z}) \cong \mathbb{Z}$  a generator. If  $W$  is an 8 dimensional smooth manifold bounding  $M$ , with orientation inducing the orientation of  $M$ , and if there exist elements  $z, c \in H^2(W, \mathbb{Z})$  such that

$$(2.1) \quad \begin{aligned} \partial W &= M, \quad z|M = u, \quad c|M = 0, \\ w_2(TW) &= c \pmod{2} \quad \text{if } M^7 \text{ of type } E_r, \\ w_2(TW) &= c + z \pmod{2} \quad \text{if } M^7 \text{ of type } \bar{E}_r, \end{aligned}$$

one defines characteristic numbers  $S_i(W, z, c) \in \mathbb{Q}$ ,  $i = 1, 2, 3$ , as follows.

$$(2.2) \quad \begin{aligned} S_1(W, z, c) &= \langle e^{\frac{c+d}{2}} \cdot \hat{A}(W), [W, \partial W] \rangle \\ S_2(W, z, c) &= \langle ch(\lambda(z) - 1) \cdot e^{\frac{c+d}{2}} \cdot \hat{A}(W), [W, \partial W] \rangle \\ S_3(W, z, c) &= \langle ch(\lambda^2(z) - 1) \cdot e^{\frac{c+d}{2}} \cdot \hat{A}(W), [W, \partial W] \rangle \\ d &= 0 \text{ if } M^7 \text{ is spin, and } d = z \text{ if } M^7 \text{ is not spin} \end{aligned}$$

Here  $\lambda(z)$  stands for the complex line bundle over  $W$  with first Chern class  $z$ ,  $ch$  is the Chern character,  $\hat{A}(W)$  the  $\hat{A}$  polynomial of  $W$ , and  $[W, \partial W]$  a fundamental class of  $W$  which, restricted to the boundary, is the fundamental class of  $M$ . The integrality of these characteristic numbers for closed manifolds, see [Hi, Theorem 26.1.1], implies that  $S_i(W, z, c) \pmod{\mathbb{Z}}$  depends only on  $\partial W = M$ , and in particular not on the choice of sign for  $u, z$  and  $c$ . Notice though that all  $S_i$  change sign, if one changes the orientation of  $M$ . Hence for manifolds of cohomology type  $E_r$  or  $\bar{E}_r$  one defines:

$$s_i(M^7) = S_i(W^8, z, c) \pmod{1}.$$

The Kreck-Stolz invariants can be interpreted as lying in  $\mathbb{Q}/\mathbb{Z}$ . In [KS1] it was shown that for any manifold of cohomology type  $E_r$  or  $\bar{E}_r$ , one can find a bounding manifold  $W$  such that (2.1) is satisfied with  $c = 0$ . In our examples we will not be able to always find an explicit bounding manifold  $W$  which is spin if  $M$  is spin, but we will be able to find a  $W$  and  $z, c$  which satisfy (2.1). M. Kreck and S. Stolz showed that the invariants  $s_i(M^7)$  are diffeomorphism invariants, and

$$\bar{s}_1(M) = 28 s_1(M) \quad \text{and} \quad \bar{s}_i(M) = s_i(M), \quad i = 2, 3$$

are homeomorphism invariants.

We now express these invariants explicitly in terms of the Pontryagin class  $p_1 = p_1(TW)$ ,  $\text{sign}(W)$ , and  $z, c$ . Recall that  $\hat{A}(W) = 1 - \frac{1}{2^3 \cdot 3} p_1 + \frac{1}{2^7 \cdot 45} (-4 p_2 + 7 p_1^2)$  and  $\text{sign}(W) = \frac{1}{45} (7 p_2 - p_1^2)$



and hence  $\hat{A}(W) = 1 - \frac{1}{2^3 \cdot 3} p_1 - \frac{1}{2^5 \cdot 7} \text{sign}(W) + \frac{1}{2^7 \cdot 7} p_1^2$ . Furthermore,  $ch(\lambda(z)) = e^z$  and hence  $ch(\lambda(z) - 1) = e^z - 1$ . Thus we obtain:

$$(2.3) \quad M^7 \text{ spin}$$

$$\begin{aligned} S_1(W, c, z) &= -\frac{1}{2^5 \cdot 7} \text{sign}(W) + \frac{1}{2^7 \cdot 7} p_1^2 - \frac{1}{2^6 \cdot 3} c^2 p_1 + \frac{1}{2^7 \cdot 3} c^4 \\ S_2(W, c, z) &= -\frac{1}{2^4 \cdot 3} z^2 p_1 + \frac{1}{2^3 \cdot 3} z^4 - \frac{1}{2^4 \cdot 3} z c p_1 + \frac{1}{2^4 \cdot 3} z c^3 + \frac{1}{2^4} z^2 c^2 + \frac{1}{2^2 \cdot 3} z^3 c \\ S_3(W, c, z) &= -\frac{1}{2^2 \cdot 3} z^2 p_1 + \frac{2}{3} z^4 - \frac{1}{2^3 \cdot 3} z c p_1 + \frac{1}{2^3 \cdot 3} z c^3 + \frac{1}{2^2} z^2 c^2 + \frac{2}{3} z^3 c. \end{aligned}$$

$$(2.4) \quad M^7 \text{ non spin}$$

$$\begin{aligned} S_1(W, c, z) &= -\frac{1}{2^5 \cdot 7} \text{sign}(W) + \frac{1}{2^7 \cdot 7} p_1^2 - \frac{1}{2^6 \cdot 3} z^2 p_1 + \frac{1}{2^7 \cdot 3} z^4 - \frac{1}{2^5 \cdot 3} z c p_1 \\ &\quad - \frac{1}{2^6 \cdot 3} c^2 p_1 + \frac{1}{2^7 \cdot 3} c^4 + \frac{1}{2^5 \cdot 3} z c^3 + \frac{1}{2^5 \cdot 3} z^3 c + \frac{1}{2^6} z^2 c^2 \\ S_2(W, c, z) &= -\frac{1}{2^3 \cdot 3} z^2 p_1 + \frac{5}{2^3 \cdot 3} z^4 - \frac{1}{2^4 \cdot 3} z c p_1 + \frac{1}{2^4 \cdot 3} z c^3 + \frac{1}{2^3} z^2 c^2 + \frac{13}{2^4 \cdot 3} z^3 c \\ S_3(W, c, z) &= -\frac{1}{2^3} z^2 p_1 + \frac{13}{2^3} z^4 - \frac{1}{2^3 \cdot 3} z c p_1 + \frac{1}{2^3 \cdot 3} z c^3 + \frac{3}{2^3} z^2 c^2 + \frac{31}{3 \cdot 2^3} z^3 c. \end{aligned}$$

These formulas need to be interpreted as follows. Since  $\partial W = M$ , we have  $H^3(\partial W, \mathbb{Q}) = H^4(\partial W, \mathbb{Q}) = 0$  and hence the inclusion  $j: (W, \emptyset) \rightarrow (W, \partial W)$  induces an isomorphism  $j^*: H^4(W, \partial W, \mathbb{Q}) \rightarrow H^4(W, \mathbb{Q})$ . Thus the characteristic classes  $p_1, z^2, c^2, zc$  in  $H^4(W, \mathbb{Q})$  can be pulled back to relative classes in  $H^4(W, \partial W, \mathbb{Q})$  and the classes  $p_1^2, z^2 p_1, z^4$ , etc. in the above formulas are abbreviations for the characteristic numbers

$$p_1^2 = \langle (j^*)^{-1}(p_1) \cup p_1, [W, \partial W] \rangle, \quad z^3 c = \langle (j^*)^{-1}(z^2) \cup zc, [W, \partial W] \rangle, \text{ etc.}$$

The main classification theorem in [KS2], Theorem 3.1, can now be stated as follows:

**THEOREM 2.5** (Kreck-Stolz). *Two simply connected smooth manifolds  $M_1, M_2$  which are both of type  $E_r$ , or both of type  $\bar{E}_r$ , are orientation preserving diffeomorphic (homeomorphic) if and only if  $s_i(M_1) = s_i(M_2)$  (resp.  $\bar{s}_i(M_1) = \bar{s}_i(M_2)$ ) for  $i = 1, 2, 3$ .*

For orientation reversing diffeomorphisms one changes the signs of the invariants.

Recall that  $E_r$  and  $\bar{E}_r$  fall into two homotopy types depending on whether  $\pi_4(M) = 0$  or  $\mathbb{Z}_2$ . For some of these manifolds B. Kruggel obtained a homotopy classification, see [Kr2, Theorem 0.1], [Kr1, Theorem 3.4]:

**THEOREM 2.6** (Kruggel). *For simply connected smooth manifolds  $M_1, M_2$  one has:*

- a) *If  $M_i$  are both of type  $E_r$  and  $\pi_4(M_i) = 0$ , then  $r$  is odd, and  $M_1$  and  $M_2$  are orientation preserving homotopy equivalent if and only if  $lk(M_1) \equiv lk(M_2) \in \mathbb{Z}_r$  and  $2r s_2(M_1) \equiv 2r s_2(M_2) \in \mathbb{Q}/\mathbb{Z}$ .*
- b) *If  $M_i$  are both of type  $E_r$  and  $\pi_4(M_i) \cong \mathbb{Z}_2$  and if  $r$  is odd,  $M_1$  and  $M_2$  are orientation preserving homotopy equivalent if and only if  $lk(M_1) \equiv lk(M_2) \in \mathbb{Z}_r$  and  $r s_2(M_1) \equiv r s_2(M_2) \in \mathbb{Q}/\mathbb{Z}$ .*
- (c) *If  $M_i$  are both of type  $\bar{E}_r$  with  $r$  divisible by 24,  $M_1$  and  $M_2$  are orientation preserving homotopy equivalent if and only if  $lk(M_1) \equiv lk(M_2) \in \mathbb{Z}_r$  and  $p_1(M_1) \equiv p_1(M_2) \pmod{24}$ .*

In the remaining cases, the homotopy classification has not yet been finished.

P. Montgantirud showed in [Mo] that in the case of manifolds of type  $\bar{E}_r$ , or of type  $E_r$  with  $r$  odd, one can replace  $s_3$  with the linking form  $lk$  in the homeomorphism and diffeomorphism classification. In other words,  $28 s_1, s_2$  and  $lk$  classify the manifold up to homeomorphism, and  $s_1, s_2$  and  $lk$  up to diffeomorphism. Furthermore, in the case of manifolds of type  $E_r$  with  $r$  odd, he proves that one can replace the invariant  $28 s_1$  with  $p_1$ .

### 3. TOPOLOGY OF SPHERE BUNDLES OVER $\mathbb{C}\mathbb{P}^2$

We start with the family of  $\mathbb{S}^3$ -bundles over  $\mathbb{C}\mathbb{P}^2$ . As we will see, the total space of such a bundle is spin if and only if the bundle itself is not spin. Our goal is to see when the total space of such a bundle is diffeomorphic to an Eschenburg space, and we thus restrict ourselves in this section to sphere bundles which are not spin. According to (1.9), they are classified by two integers  $a, b$  and we let

$$\mathbb{S}^3 \rightarrow S_{a,b} \xrightarrow{\pi} \mathbb{C}\mathbb{P}^2 \quad \text{with} \quad p_1(S_{a,b}) = (2a + 2b + 1)x^2, \quad e(S_{a,b}) = (a - b)x^2 \quad \text{and} \quad w_2 \neq 0$$

where  $x$  is a generator of  $H^2(\mathbb{C}\mathbb{P}^2; \mathbb{Z})$ . A change of orientation corresponds to changing the sign of  $e$  but not of  $p_1$ . Thus  $S_{a,b}$  and  $S_{b,a}$  are orientation reversing diffeomorphic.

It turns out that it is also the non-spin bundles which are known to admit non-negative curvature, as was shown in [GZ2]:

**THEOREM 3.1** (Grove-Ziller). *Every sphere bundle over  $\mathbb{C}\mathbb{P}^2$  with  $w_2 \neq 0$  admits a metric with non-negative sectional curvature.*

As for their topology we have:

**PROPOSITION 3.2.** *The manifolds  $S_{a,b}$  have cohomology type  $E_r$  with  $r = |a - b|$ , as long as  $a \neq b$ . Their first Pontryagin class is given by  $p_1(TS_{a,b}) \equiv (2a + 2b + 4) \pmod{r}$ .*

*Proof.* The cohomology ring structure and  $r = |a - b|$  immediately follows from the Gysin sequence. In particular,  $H^2(S_{a,b}, \mathbb{Z}) \cong \mathbb{Z}$  with  $u = \pi^*(x)$  a generator.

For the manifolds  $S = S_{a,b}$  we have  $TS \cong \pi^*(T\mathbb{C}\mathbb{P}^2) \oplus V$  where  $V \oplus \text{Id} = \pi^*E$  with  $\text{Id}$  a trivial bundle and  $E$  the vector bundle associated to the sphere bundle  $S$ . Hence

$p_1(V) = \pi^*(p_1(E))$  and, since  $p_1(T\mathbb{C}\mathbb{P}^2) = 3x^2$  and  $p_1(S) = (2a + 2b + 1)x^2$ , we have  $p_1(TS) = (2a + 2b + 4)u^2$ . Also,  $S$  is spin since  $w_2(TS) = \pi^*(w_2(T\mathbb{C}\mathbb{P}^2) + w_2(E)) = 0$ .  $\square$

*Remark.* In the case of  $a = b$  the bundle has the same cohomology ring as  $\mathbb{S}^3 \times \mathbb{C}\mathbb{P}^2$ . Such manifolds are not classified by the Kreck-Stolz invariants. However, they are also not homotopy equivalent to any known example of positive curvature.

We now compute the Kreck-Stolz invariants and for this purpose need to fix the orientation. In the fibration of  $S_{a,b}$ , the base  $\mathbb{C}\mathbb{P}^2$  is oriented via  $x^2 \in H^4(\mathbb{C}\mathbb{P}^2)$  and the orientation of the fiber is determined by the sign of the Euler class. This determines the orientation on the total space.

**PROPOSITION 3.3.** *The Kreck-Stolz invariants for  $S_{a,b}$  with  $a \neq b$  are given by:*

$$\begin{aligned} s_1(S_{a,b}) &\equiv \frac{1}{2^5 \cdot 7 \cdot (a-b)} (a+b+2)^2 - \frac{\text{sgn}(a-b)}{2^5 \cdot 7} \pmod{1} \\ s_2(S_{a,b}) &\equiv \frac{-1}{2^3 \cdot 3 \cdot (a-b)} (a+b+1) \pmod{1} \\ s_3(S_{a,b}) &\equiv \frac{-1}{2 \cdot 3 \cdot (a-b)} (a+b-2) \pmod{1} \end{aligned}$$

*Proof.* We will use the notation established in the proof of (3.2) and set  $u = \pi^*(x)$ . Let  $E$  be the vector bundle associated to the sphere bundle  $S_{a,b}$  and  $\bar{\pi}: W_{a,b} \rightarrow \mathbb{C}\mathbb{P}^2$  its disk bundle. Hence  $W = W_{a,b}$  is a natural choice for a bounding manifold and we identify the cohomology of  $W$  with that of  $\mathbb{C}\mathbb{P}^2$ . Thus  $z := \bar{\pi}^*(x)$  is a generator for  $H^2(W; \mathbb{Z}) \cong \mathbb{Z}$ . Since the restriction of  $\bar{\pi}$  to  $\partial W = S_{a,b}$  is  $\pi$ , it follows that  $z|\partial W = \pi^*(x) = u$ . For the tangent bundle of  $W$  we have  $TW = \bar{\pi}^*(T\mathbb{C}\mathbb{P}^2) \oplus \bar{\pi}^*E$  and hence  $w_2(TW) = \bar{\pi}^*(w_2(T\mathbb{C}\mathbb{P}^2) + w_2(E)) = 0$ . Thus  $S_{a,b}$  and  $W$  are both spin, and we can choose  $c = 0$  in (2). Furthermore,  $p_1(TW) = \bar{\pi}^*(p_1(T\mathbb{C}\mathbb{P}^2) + p_1(E)) = 3x^2 + (2a + 2b + 1)x^2 = (2a + 2b + 4)z^2 \in H^4(W; \mathbb{Z}) \cong \mathbb{Z}$ .

The sphere bundle and hence the disk bundle are assumed to be oriented and we let  $U \in H^4(W, \partial W) \cong \mathbb{Z}$  be the corresponding Thom class. The orientation on  $\mathbb{C}\mathbb{P}^2$  is defined by  $\langle x^2, [\mathbb{C}\mathbb{P}^2] \rangle = 1$  and we define the orientation on  $W$  such that  $U \cap [W, \partial W] = [\mathbb{C}\mathbb{P}^2]$ . Thus  $\langle U \cup z^2, [W, \partial W] \rangle = \langle z^2, U \cap [W, \partial W] \rangle = \langle x^2, [\mathbb{C}\mathbb{P}^2] \rangle = 1$ . If  $j: W \rightarrow (W, \partial W)$  is the inclusion,  $j^*(U)$  is the Euler class and hence  $j^*(U) = (a-b)z^2$ . Thus we obtain

$$\begin{aligned} \langle (j^{-1})^*(z^2) \cup z^2, [W, \partial W] \rangle &= \frac{1}{a-b} \langle U \cup z^2, [W, \partial W] \rangle = \frac{1}{a-b} \\ \langle (j^{-1})^*(p_1) \cup p_1, [W, \partial W] \rangle &= \frac{1}{a-b} \langle (2a + 2b + 4)^2 \cdot U \cup z^2, [W, \partial W] \rangle \\ &= \frac{1}{a-b} (2a + 2b + 4)^2 \\ \langle (j^{-1})^*(p_1) \cup z^2, [W, \partial W] \rangle &= \frac{1}{a-b} \langle (2a + 2b + 4) \cdot U \cup z^2, [W, \partial W] \rangle \\ &= \frac{1}{a-b} (2a + 2b + 4). \end{aligned}$$

Recall that the signature of a manifold with boundary is defined as the signature of the quadratic form on  $H^4(W, \partial W)$  given by  $v \mapsto \langle j^*(v) \cup v, [W, \partial W] \rangle$ . Since  $H^4(W, \partial W)$  is generated by  $U$  and  $\langle (j^*)(U) \cup U, [W, \partial W] \rangle = a - b$  we have  $\text{sign}(W) = \text{sgn}(a - b)$ . Substituting into (2) proves our claim.  $\square$

*Remark.* Notice that  $s_3 \equiv 4s_2 + \frac{1}{2r} \pmod{1}$  and thus for (orientation preserving diffeomorphisms)  $s_2$  determines  $s_3$ .

If  $r > 1$  we also have the linking form:

**COROLLARY 3.4.** *The linking form of  $S_{a,b}$  with  $a \neq b$  is standard, i.e.,  $lk(S_{a,b}) \equiv \frac{1}{(a-b)} \in \mathbb{Q}/\mathbb{Z}$ .*

*Proof.* As discussed in Section 1, the linking form is a bilinear form  $L : H^4(M; \mathbb{Z}) \times H^4(M; \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$  and is completely determined by  $L(u^2, u^2)$ . It turns out by [KS2], see also [Mo], that for manifolds of type  $E_r$  we can also express the linking form as the characteristic number  $z^4$ . As seen above in the case of  $S_{a,b}$  we obtain  $z^4 = \frac{1}{a-b}$  and hence  $L(u^2, u^2) \equiv \frac{1}{a-b} \in \mathbb{Q}/\mathbb{Z}$ .  $\square$

As a consequence of Proposition 3.3, together with Theorem 2.5, one easily obtains a homeomorphism and diffeomorphism classification of the manifolds  $S_{a,b}$ :

**COROLLARY 3.5.** *The manifolds  $S_{a,b}$  and  $S_{a',b'}$  with  $r = a - b = a' - b' > 0$  are*

- (a) *orientation preserving homeomorphic if and only if  $a \equiv a' \pmod{12r}$ .*
- (b) *orientation preserving diffeomorphic if and only if*

$$a \equiv a' \pmod{12r} \quad \text{and} \quad (a - a') [a + a' - r + 2] \equiv 0 \pmod{2^3 \cdot 7 \cdot r}.$$

- (a') *orientation reversing homeomorphic if and only if  $r = 1$  and  $a \equiv -a' \pmod{12}$ .*
- (b') *orientation reversing diffeomorphic if and only if  $r = 1$  and  $a(a+1) \equiv -a'(a'+1) \pmod{2^3 \cdot 7}$ .*

*Remark.* (a) Notice that if  $a \equiv a' \pmod{168r}$ , then  $S_{a,r-a}$  and  $S_{a',r-a'}$  are diffeomorphic. Thus each sphere bundle is diffeomorphic to infinitely many other sphere bundles.

(b) As far as the homotopy type is concerned, we note that  $\pi_4(S_{a,b}) \cong \mathbb{Z}_2$  if  $r = |a - b|$  is even, as follows from Theorem 2.6 (a). We suspect that if  $r$  is odd,  $\pi_4(S_{a,b}) = 0$ . For example, Corollary 5.10 implies that  $\pi_4(S_{\pm 1, p(p+1)}) = 0$ .

In any case, if  $\pi_4(S_{a,b}) = 0$ ,  $S_{a,b}$  and  $S_{a',b'}$  with  $a - b = a' - b' > 0$  are orientation preserving homotopy equivalent if and only if  $a \equiv a' \pmod{6}$ , and orientation reversing homotopy equivalent if and only if  $a + a' \equiv r - 1 \pmod{6}$ . If  $\pi_4(S_{a,b}) \cong \mathbb{Z}_2$ , the same holds mod 12.

#### 4. TOPOLOGY OF CIRCLE BUNDLES

In this section we discuss the manifolds  $M_{a,b}^t$  described in the introduction. We start with 6-dimensional manifolds which are  $\mathbb{S}^2$  bundles over  $\mathbb{C}\mathbb{P}^2$ . As we will see, it is again

important to assume that they are not spin. According to (1.8), the corresponding  $\mathrm{SO}(3)$  principal bundle satisfies  $p_1 \equiv 1 \pmod{4}$  and we define:

$$\mathbb{S}^2 \rightarrow N_t \xrightarrow{\pi} \mathbb{C}\mathbb{P}^2 \quad \text{with} \quad p_1(N_t) = (1 - 4t)x^2 \text{ and } w_2 \neq 0$$

for some integer  $t$ .

In order to compute the cohomology ring of  $N_t$  we regard the 2-sphere bundle as the projectivization of a rank 2 complex vector bundle. For this purpose, let  $\mathrm{U}(2) \rightarrow P \rightarrow \mathbb{C}\mathbb{P}^2$  be a principal  $\mathrm{U}(2)$  bundle with associated vector bundle  $E = P \times_{\mathrm{U}(2)} \mathbb{C}^2$ . Such bundles are classified by their Chern classes  $c_1$  and  $c_2$  and we define

$$\mathrm{U}(2) \rightarrow P_t \xrightarrow{\tau} \mathbb{C}\mathbb{P}^2 \quad \text{with} \quad c_1(P_t) = x \text{ and } c_2(P_t) = tx^2$$

Since  $c_1 \pmod{2} = w_2$ , such a bundle is not spin. If  $Z = \mathrm{diag}(z, z) \subset \mathrm{U}(2)$  denotes the center of  $\mathrm{U}(2)$ , we obtain an  $\mathrm{SO}(3)$  principal bundle

$$\mathrm{U}(2)/Z \cong \mathrm{SO}(3) \rightarrow Q_t := P_t/Z \rightarrow \mathbb{C}\mathbb{P}^2.$$

According to (1.6), we have  $P_t/Z = P_-$  and hence, by (1.5),  $p_1(Q_t) = c_1^2 - 4c_2 = (1 - 4t)x^2$ . Furthermore,  $w_2(Q_t) = w_2(P_-) = w_2(P) \neq 0$ . Thus

$$\mathrm{SO}(3) \rightarrow Q_t \rightarrow \mathbb{C}\mathbb{P}^2 \quad \text{with} \quad p_1(Q_t) = (1 - 4t)x^2 \text{ and } w_2(Q_t) \neq 0$$

is the  $\mathrm{SO}(3)$  principal bundle associated to the 2-sphere bundle  $N_t$ . This implies that

$$P(E) \simeq P_t \times_{\mathrm{U}(2)} \mathbb{C}\mathbb{P}^1 \simeq (P_t/Z) \times_{\mathrm{SO}(3)} \mathbb{S}^2 \simeq Q_t \times_{\mathrm{SO}(3)} \mathbb{S}^2 \simeq N_t$$

i.e., we can regard  $N_t$  as the projectivization of  $E$ . Furthermore,  $N_t = P_t/\mathrm{T}^2$  since

$$N_t \simeq P(E) \simeq P_t \times_{\mathrm{U}(2)} \mathbb{C}\mathbb{P}^1 \simeq P_t \times_{\mathrm{U}(2)} \mathrm{U}(2)/\mathrm{T}^2 \simeq P_t/\mathrm{T}^2.$$

We can now apply Leray-Hirsch to compute the cohomology ring of  $P(E) = N_t$ :

$$\begin{aligned} H^*(P(E)) &\cong H^*(\mathbb{C}\mathbb{P}^2)[y]/[y^2 + c_1(E)y + c_2(E)] \\ (4.1) \quad &\cong H^*(\mathbb{C}\mathbb{P}^2)[y]/[y^2 + xy + tx^2] \\ &\cong \mathbb{Z}[x, y]/[x^3 = 0, y^2 + xy + tx^2 = 0] \end{aligned}$$

Here we have identified  $x \in H^2(\mathbb{C}\mathbb{P}^2)$  with  $\pi^*(x) \in H^2(N_t) \cong H^2(P(E))$ . Furthermore, the generator  $y$  is defined by  $y = c_1(S^*)$  where  $S^*$  is the dual of the tautological complex line bundle  $S$  over  $P(E)$ . Hence

$$\begin{aligned} H^2(N_t) &\cong \mathbb{Z} \oplus \mathbb{Z} \text{ with generators } x, y; \\ (4.2) \quad H^4(N_t) &\cong \mathbb{Z} \oplus \mathbb{Z} \text{ with generators } x^2, xy, \text{ and relationship } y^2 = -xy - tx^2; \\ H^6(N_t) &\cong \mathbb{Z} \text{ with generator } x^2y, \text{ and } x^3 = 0, y^2x = -x^2y, y^3 = (1 - t)x^2y. \end{aligned}$$

Notice that, since the quadratic relationship  $y^2 + xy + tx^2$  has discriminant  $t - \frac{1}{4}$ , the manifolds  $N_t$  all have different homotopy type.

We now consider circle bundles over  $N_t$ . They are classified by their Euler class  $e \in H^2(N_t, \mathbb{Z})$ . For symmetry reasons, see Corollary 5.3, we define

$$\mathbb{S}^1 \rightarrow M_{a,b}^t \xrightarrow{\sigma} N_t \quad \text{with} \quad e(M_{a,b}^t) = ax + (a + b)y$$

where  $a, b$  are arbitrary integers. In order to ensure that  $M_{a,b}^t$  is simply connected we assume that  $(a, b) = 1$ . The total space is oriented via the orientation class  $x^2y \in H^6(N_t)$  on the base and the orientation on the fiber given by  $e$ . Thus  $M_{a,b}^t$  and  $M_{-a,-b}^t$  are orientation reversing diffeomorphic.

For the basic topological invariants of  $M_{a,b}^t$  we obtain:

**PROPOSITION 4.3.** *The manifolds  $M_{a,b}^t$  have cohomology type  $E_r$  with  $r = |t(a+b)^2 - ab|$ , as long as  $t(a^2 + b^2) \neq ab$ . Furthermore, the first Pontryagin class is given by  $p_1(TM_{a,b}^t) \equiv 4(1-t)(a+b)^2 \pmod{r}$ .*

*Proof.* The Gysin sequence for the bundle  $M_{a,b}^t \rightarrow N_t$  yields  $H^2(M_{a,b}^t; \mathbb{Z}) \cong H^5(M_{a,b}^t; \mathbb{Z}) \cong \mathbb{Z}$ ;  $H^1(M_{a,b}^t; \mathbb{Z}) \cong H^3(M_{a,b}^t; \mathbb{Z}) \cong H^6(M_{a,b}^t; \mathbb{Z}) \cong 0$  and  $H^4(M_{a,b}^t; \mathbb{Z}) \cong \mathbb{Z}_r$  where  $\mathbb{Z}_r$  is the cokernel of

$$0 \longrightarrow H^2(N_t; \mathbb{Z}) \xrightarrow{\cup e} H^4(N_t; \mathbb{Z}) \longrightarrow H^4(M_{a,b}^t; \mathbb{Z}) \rightarrow 0.$$

Since the cup product with the Euler class sends  $x$  to  $ax^2 + (a+b)xy$  and  $y$  to  $axy + (a+b)y^2 = -t(a+b)x^2 - bxy$  we obtain for this homomorphism that

$$\det \begin{pmatrix} a & -t(a+b) \\ a+b & -b \end{pmatrix} = t(a+b)^2 - ab$$

and thus the cokernel has order  $|t(a+b)^2 - ab|$  as long as  $t(a+b)^2 - ab \neq 0$ . One easily sees that the cokernel is cyclic since  $(a, b) = 1$ , and hence  $H^4(M_{a,b}^t; \mathbb{Z}) \cong \mathbb{Z}_r$  with  $r = |t(a+b)^2 - ab|$ .

To compute the characteristic classes we observe that the tangent bundles split:  $TM_{a,b}^t = \sigma^*(TN_t) \oplus \text{Id}$  and  $TN_t = \pi^*(T\mathbb{C}P^2) \oplus V$  with  $V \oplus \text{Id} = \pi^*(E^3)$ , and where  $E^3$  is the rank 3 vector bundle corresponding to the 2-sphere bundle  $N_t$ . Hence  $p_1(TM_{a,b}^t) = \sigma^*\pi^*(3x^2 + (1-4t)x^2)$ . Applying the Gysin sequence again, we see that  $0 = \sigma^*(e) = \sigma^*(ax + (a+b)y)$  and that  $\sigma^*: H^2(N_t; \mathbb{Z}) \rightarrow H^2(M_{a,b}^t; \mathbb{Z})$  is onto. Since  $(a, b) = 1$ , this implies that there exists a generator  $u \in H^2(M_{a,b}^t; \mathbb{Z}) \cong \mathbb{Z}$  with  $\sigma^*(x) = -(a+b)u$  and  $\sigma^*(y) = au$ . Hence  $p_1(TM_{a,b}^t) = 4(1-t)(a+b)^2u^2$  and similarly  $M_{a,b}^t$  is spin since  $w_2(TM_{a,b}^t) = \sigma^*(w_2(TN_t))$  and  $w_2(TN_t) = \pi^*(2x) = 0$ .  $\square$

The Kreck-Stolz invariants for the manifolds  $M_{a,b}^t$  were computed in the special cases  $t = \pm 1$  in [KS1] and [AMP1]. For the general case below, we will use  $s = t(a+b)^2 - ab$ ,  $r = |s|$ , since  $s$  is not always positive.

**PROPOSITION 4.4.** *If  $s = t(a+b)^2 - ab \neq 0$ , the Kreck-Stolz invariants for  $M_{a,b}^t$  are given by:*

$$s_1(M_{a,b}^t) \equiv \frac{-1}{2^5 \cdot 7} \text{sign}(W) - \frac{(a+b)(t-1)^2}{2^3 \cdot 7 \cdot s} + \frac{a+b}{2^5 \cdot 3 \cdot 7} \{3ab + (t-1)(8 + (a+b)^2)\} \pmod{1},$$

$$\begin{aligned}
 s_2(M_{a,b}^t) &\equiv \frac{1}{2^3 \cdot 3} \left\{ (t-1)(m+n) [2 - (a+b)(m+n) - 2(m+n)^2] \right. \\
 &\quad \left. - am(m+2n) - bn(n+2m) - 6mn(m+n) \right\} \\
 &\quad + \frac{1}{2^3 \cdot 3 \cdot s} \left\{ (t^2-1)(a+b)(n+m)^2(2 - (n+m)^2) \right. \\
 &\quad \left. + (t-1) [m^4(3a+b) + n^4(a+3b) - 2(a+b)(m+n)^2 + 2(am^2 + bn^2)(2nm-1)] \right. \\
 &\quad \left. + am^4 + bn^4 - 6m^2n^2(a+b) - 4mn(an^2 + bm^2) \right\} \pmod{1},
 \end{aligned}$$

$$\begin{aligned}
 s_3(M_{a,b}^t) &\equiv \frac{1}{2 \cdot 3} \left\{ (t-1)(m+n) [1 - (a+b)(m+n) - 4(m+n)^2] \right. \\
 &\quad \left. - am(m+2n) - bn(n+2m) \right\} \\
 &\quad + \frac{1}{3 \cdot s} \left\{ (t^2-1)(a+b)(n+m)^2(1 - 2(n+m)^2) \right. \\
 &\quad \left. + (t-1) [2m^4(3a+b) + 2n^4(a+3b) - (a+b)(m+n)^2 + (am^2 + bn^2)(8nm-1)] \right. \\
 &\quad \left. + 2am^4 + 2bn^4 - 12m^2n^2(a+b) - 8mn(an^2 + bm^2) \right\} \pmod{1},
 \end{aligned}$$

where  $n, m \in \mathbb{Z}$  are chosen such that  $am - bn = 1$ . Furthermore,

$$\text{sign}(W) = \begin{cases} 0, & \text{if } s > 0 \\ 2, & \text{if } s < 0 \text{ and } b + (1-t)(a+b) > 0 \\ -2, & \text{if } s < 0 \text{ and } b + (1-t)(a+b) < 0. \end{cases}$$

*Proof.* A natural choice for a bounding manifold is the disk bundle  $\sigma': W_{a,b}^8 \rightarrow N_t$  of the rank 2 vector bundle  $E^2$  associated to the circle bundle  $\sigma$ . Recall from the proof of Proposition 4.3 that there exists a generator  $u \in H^2(M_{a,b}^t; \mathbb{Z}) \cong \mathbb{Z}$  with  $\sigma^*(x) = -(a+b)u$  and  $\sigma^*(y) = au$ . Furthermore, we identified the cohomology of  $W$  with that of  $N_t$  via  $\sigma'^*$ . In order to apply (2), we need to choose classes  $z, c \in H^2(W, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$  with  $z|\partial W = u$ ,  $c|\partial W = 0$  and  $w_2(W) \equiv c \pmod{2}$ . If we choose integers  $m, n$  with  $am - bn = 1$  and set  $z = nx + (m+n)y$  and  $c = e(M_{a,b}^t) = ax + (a+b)y$ , it follows that  $\sigma^*(z) = u$ ,  $\sigma^*(c) = 0$ . Furthermore,  $w_2(TW) = \sigma'^*(w_2(TN_t)) + w_2(E^2) = e \pmod{2}$ . Thus  $z$  and  $c$  have the properties required in (2.1). Using  $p_1(TW) = \sigma'^*(p_1(TN_t)) + p_1(E^2) = 4(1-t)x^2 + e^2$  and replacing the value of  $p_1$  in (2) we obtain

$$\begin{aligned}
 S_1(W, c, z) &= -\frac{1}{2^5 \cdot 7} \text{sign}(W) + \frac{1}{2^5 \cdot 3 \cdot 7} (12(t-1)^2 x^4 + 8(t-1)x^2 c^2 - c^4), \\
 S_2(W, c, z) &= \frac{1}{2^3 \cdot 3} (2(t-1)zcx^2 + 2(t-1)z^2x^2 + z^2c^2 + 2z^3c + z^4), \\
 S_3(W, c, z) &= \frac{1}{2 \cdot 3} ((t-1)zcx^2 + 2(t-1)z^2x^2 + z^2c^2 + 4z^3c + 4z^4).
 \end{aligned}$$

Recall that the orientation for  $N_t$  is chosen so that  $\langle x^2 y, [N_t] \rangle = 1$ . The orientation for the vector bundle  $E^2$  defines a Thom class  $U \in H^2(W, \partial W) \cong \mathbb{Z}$  and we define the orientation on  $W$  such that  $U \cap [W, \partial W] = [N_t]$ .

We first compute the characteristic numbers involving  $c$  by using the fact that  $j^*(U) = e(E^2) = c$  where  $j : W \rightarrow (W, \partial W)$ :

$$\begin{aligned}
c^4 &= \langle c^3 \cup (j^*)^{-1}(c), [W, \partial W] \rangle = \langle c^3 \cup U, [W, \partial W] \rangle \\
&= \langle (ax + (a+b)y)^3, [N_t] \rangle = a^3 + b^3 - t(a+b)^3 \\
z^2 c^2 &= \langle c z^2 \cup U, [W, \partial W] \rangle = \langle (ax + (a+b)y)(nx + (m+n)y)^2, [N_t] \rangle \\
&= an^2 + bm^2 - t(a+b)(m+n)^2 \\
z^3 c &= \langle z^3 \cup U, [W, \partial W] \rangle = \langle (nx + (m+n)y)^3, [N_t] \rangle \\
&= m^3 + n^3 - t(m+n)^3 \\
x^2 c^2 &= \langle x^2 c \cup U, [W, \partial W] \rangle = \langle x^2(ax + (a+b)y), [N_t] \rangle = a+b \\
zc x^2 &= \langle z x^2 \cup U, [W, \partial W] \rangle = \langle (nx + (m+n)y)x^2, [N_t] \rangle = m+n
\end{aligned}$$

A calculation shows that  $s x^2 = -(bx + (a+b)y) \cup c$  and hence

$$x^4 = -\frac{1}{s} \langle x^2(bx + (a+b)y), [N_t] \rangle = -\frac{a+b}{s}$$

Similarly,  $z^2 = (\alpha x + \beta y) \cup c$ , where  $s\alpha = at(n^2 - m^2) + 2btn(m+n) - n^2b$  and  $s\beta = t(a+b)(m+n)^2 - am^2 - bn^2$ .

Thus

$$\begin{aligned}
z^4 &= \langle z^2 \cup (\alpha x + \beta y) \cup U, [W, \partial W] \rangle = \langle (nx + (m+n)y)^2 \cup (\alpha x + \beta y), [N_t] \rangle \\
&= \frac{1}{s} \{ -t^2(a+b)(m+n)^4 + t[m^4(3a+b) + n^4(a+3b) + 4nm(am^2 + bn^2)] \\
&\quad - am^4 - bn^4 \} \\
z^2 x^2 &= \frac{1}{s} \langle (\alpha x + \beta y)x^2, [N_t] \rangle = \frac{1}{s} (t(a+b)(m+n)^2 - am^2 - bn^2)
\end{aligned}$$

We now compute the signature form  $v \rightarrow \langle j^*(v) \cup v, [W, \partial W] \rangle$  on  $H^4(W, \partial W)$ . Since  $x, y$  are a basis of  $H^2(W) \cong H^2(N_t)$ , the classes  $x \cup U, y \cup U$  are a basis of  $H^4(W, \partial W)$ . Using  $j^*(x \cup U) = x \cup j^*(U) = x \cup c = ax^2 + (a+b)xy$  and similarly  $j^*(y \cup U) = -t(a+b)x^2 - bxy$ , we have

$$\begin{aligned}
\langle j^*(x \cup U) \cup x \cup U, [W, \partial W] \rangle &= \langle (ax^2 + (a+b)xy) \cup x, [N_t] \rangle = a+b \\
\langle j^*(x \cup U) \cup y \cup U, [W, \partial W] \rangle &= \langle (ax^2 + (a+b)xy) \cup y, [N_t] \rangle = -b \\
\langle j^*(y \cup U) \cup x \cup U, [W, \partial W] \rangle &= \langle (-t(a+b)x^2 - bxy) \cup x, [N_t] \rangle = -b \\
\langle j^*(y \cup U) \cup y \cup U, [W, \partial W] \rangle &= \langle (-t(a+b)x^2 - bxy) \cup y, [N_t] \rangle = -t(a+b) + b
\end{aligned}$$

If we denote by  $R$  the matrix of this signature form, one easily sees that  $\det R = -s$  and since  $\text{tr } R = b + (1-t)(a+b)$ , the signature is as claimed.

Combining all of the above, our claim follows from (2).  $\square$

*Remark.* Notice that one can always choose  $m$  or  $n$  to be divisible by 4, which easily implies that the term  $(t-1)(m+n)[2-2(m+n)^2] - 6mn(m+n)$  vanishes in  $s_2$  and the term  $(t-1)(m+n)[1-4(m+n)^2] \equiv 3 \pmod{6}$  in  $s_3$ . Thus a change  $(a, b, m, n) \rightarrow (b, a, -n, -m)$  gives the same Kreck-Stolz invariants, confirming the orientation preserving diffeomorphism in Corollary 5.3. Notice also that a change  $(a, b, m, n) \rightarrow (-a, -b, -m, -n)$  gives



opposite Kreck-Stolz invariants confirming that  $M_{a,b}^t$  is orientation reversing diffeomorphic to  $M_{-a,-b}^t$ .

Recall that for manifolds of type  $E_r$  with  $r > 1$  the linking form is equal to the characteristic number  $z^4$  and we hence obtain:

**COROLLARY 4.5.** *If  $s = t(a+b)^2 - ab$  and  $r = |s| > 1$ , the linking form of  $M_{a,b}^t$  is given by*

$$lk(M_{a,b}^t) = \frac{1}{s} \{-t^2(a+b)(m+n)^4 + t[m^4(3a+b) + n^4(a+3b) + 4nm(am^2 + bn^2)] - am^4 - bn^4\}$$

in  $\mathbb{Q}/\mathbb{Z}$ , where  $n, m \in \mathbb{Z}$  are chosen such that  $am - bn = 1$ .

*Remark.* As far as the homotopy invariant  $\pi_4(M_{a,b}^t)$  is concerned, we notice that it does not depend on  $a, b$  as follows from the circle bundle  $S^1 \rightarrow P_t \rightarrow M_{a,b}^t$ . This implies that  $\pi_4(M_{a,b}^t) = \mathbb{Z}_2$  for  $t$  even. Indeed, if  $t$  even,  $a$  even and  $b$  odd, the order  $r = |t(a+b)^2 - ab|$  is even and the claim follows from Theorem 2.6 (a). If  $t$  is odd, we suspect that  $\pi_4(M_{a,b}^t) = 0$ . For example, from Proposition 5.4 it follows that  $\pi_4(M_{a,b}^{\pm 1}) = 0$ .

## 5. GEOMETRY OF SPHERE BUNDLES AND CIRCLE BUNDLES

In this section we study the geometry of the sphere and circle bundles defined in Section 3 and 4 and the relationships between them.

We remind the reader of the various bundle structures in Section 3 and 4 in the following diagram:

$$\begin{array}{ccccc}
 S^1 & & S^1 & & \\
 \downarrow & & \downarrow & & \\
 U(2) & \longrightarrow & P_t & \xrightarrow{\tau} & \mathbb{C}\mathbb{P}^2 \\
 \downarrow /Z & & \downarrow /Z & & \\
 SO(3) & \longrightarrow & Q_t & \longrightarrow & \mathbb{C}\mathbb{P}^2 \\
 \\ 
 \mathbb{C}\mathbb{P}^1 & \longrightarrow & N_t & \xrightarrow{\pi} & \mathbb{C}\mathbb{P}^2 \\
 \\ 
 T^2 & & S^1 & & S^1 \\
 \searrow & & \downarrow & & \downarrow \\
 S^1 & \longrightarrow & P_t & \longrightarrow & M_{a,b}^t \\
 & & \downarrow & \searrow & \downarrow \sigma \\
 S^1 & \longrightarrow & Q_t & \longrightarrow & N_t \\
 \\ 
 S^3 & \longrightarrow & S_{a,b} & \xrightarrow{\pi} & \mathbb{C}\mathbb{P}^2
 \end{array}$$

In particular, recall that we can regard the 2-sphere bundle  $N_t$  as the projectivization of a rank 2 complex vector bundle  $P_t \times_{U(2)} \mathbb{C}^2$  with  $c_1(P_t) = x$ ,  $c_2(P_t) = tx^2$  and that  $N_t = P_t/T^2$  as well. We defined  $M_{a,b}^t$  as the circle bundle over  $N_t$  with Euler class  $e = ax + (a+b)y$ .

*A different description of  $M_{a,b}^t$ .*

PROPOSITION 5.1. *The circle bundle  $S^1 \rightarrow M_{a,b}^t \rightarrow N_t$  can be equivalently described as the circle bundle  $T^2/S_{a,b}^1 \rightarrow P_t/S_{a,b}^1 \rightarrow P_t/T^2$ , where  $S_{a,b}^1 = \text{diag}(z^a, z^b) \subset T^2 \subset U(2)$ .*

*Proof.* We first claim that, using the basis  $x, y \in H^2(P_t/T^2)$  from Section 4, the first Chern class  $c_1(P_t/S_{a,b}^1) = rx + sy$  for some functions  $r, s$  linear in  $a, b$ . To see this, observe that  $H^*(P_t) \cong H^*(\mathbb{S}^3 \times \mathbb{S}^5)$  since in the spectral sequence of the principal bundle  $U(2) \rightarrow P_t \rightarrow \mathbb{C}\mathbb{P}^2$ , the differential  $d_2: H^1(U(2)) \rightarrow H^2(\mathbb{C}\mathbb{P}^2)$  takes a generator to  $c_1(P_t) = x$ . This holds for any  $U(2)$  principal bundle, as can be seen by observing that this is true in the universal bundle and hence via pullback for any  $U(2)$ -bundle. Now consider the commutative diagram of fibrations:

$$\begin{array}{ccccccc} T^2 & \longrightarrow & P_t & \longrightarrow & P_t/T^2 & \xrightarrow{g_1} & B_{T^2} \\ \downarrow f & & \downarrow & & \downarrow id & & \downarrow B_f \\ S^1 \simeq T^2/S_{a,b}^1 & \longrightarrow & P_t/S_{a,b}^1 & \xrightarrow{h} & P_t/T^2 & \xrightarrow{g_2} & B_{S^1} \end{array}$$

where  $g_1$  and  $g_2$  are the classifying maps of the respective  $T^2$  bundle and  $S^1$  bundle. If we choose bases  $\lambda \in H^1(S^1)$  and  $\mu, \nu \in H^1(T^2)$ , the projection  $f$  induces a map  $f^*: H^1(S^1) \rightarrow H^1(T^2)$  with  $f^*(\lambda) = \tilde{r}\mu + \tilde{s}\nu$  for some functions  $\tilde{r}, \tilde{s}$  linear in  $a, b$ . Thus via transgression  $H^1(S^1) \cong H^2(B_{S^1})$  and  $H^1(T^2) \cong H^2(B_{T^2})$  it follows that  $B_f^*(\bar{\lambda}) = \tilde{r}\bar{\mu} + \tilde{s}\bar{\nu}$ . From the spectral sequence of  $P_t \rightarrow P_t/T^2 \rightarrow B_{T^2}$  it follows that  $g_1^*$  is an isomorphism in  $H^2$ . Since  $g_2^*$  takes the canonical generator in  $H^2(B_{S^1}) \cong \mathbb{Z}$  to  $c_1(P_t/S_{a,b}^1)$ , it follows that  $c_1(P_t/S_{a,b}^1) = \tilde{r}g_1^*(\bar{\mu}) + \tilde{s}g_1^*(\bar{\nu})$ . Via a basis change the claim follows.

We now show that  $c_1(P_t/S_{0,1}^1) = y$  and  $c_1(P_t/S_{1,-1}^1) = x$  and thus  $c_1(P_t/S_{a,b}^1) = ax + (a+b)y$  which implies that  $M_{a,b}^t = P_t/S_{a,b}^1$ . To evaluate the two Euler classes, we need to specify the orientation of the circle bundle  $T^2/S_{a,b}^1 \rightarrow P_t/S_{a,b}^1 \rightarrow P_t/T^2$ . If the Lie algebra of  $T^2$  is endowed with its natural orientation, the Lie algebras of  $S_{a,b}^1$  and  $S_{-b,a}^1$  form an oriented basis. Thus the action of  $\text{diag}(e^{-b\theta}, e^{a\theta}) \subset U(2)$  on  $P_t$  induces a (possibly ineffective) circle action on  $P_t/S_{a,b}^1$ , which is the orientation we will use in the following.

To see that  $c_1(P_t/S_{0,1}^1) = y$ , recall that  $y = c_1(S^*)$  where  $S$  is the canonical line bundle over  $P(E) = P_t \times_{U(2)} \mathbb{C}\mathbb{P}^1$ . Consider the circle bundle

$$S^1 \simeq T^2 / \text{diag}(1, z) \rightarrow U(2) / \text{diag}(1, z) \simeq \mathbb{S}^3 \rightarrow U(2) / T^2 \simeq \mathbb{C}\mathbb{P}^1.$$

We identify  $U(2)/\text{diag}(1, z)$  with  $\mathbb{S}^3$  by sending  $A \in U(2)$  to its first column vector. Thus the action of  $\text{diag}(e^{i\theta}, 1)$  on  $U(2)/\text{diag}(1, z)$  from the right is multiplication by  $e^{i\theta}$  in both coordinates and hence the Hopf action, which shows that this is the canonical line bundle over  $\mathbb{C}\mathbb{P}^1$ . It follows that

$$T^2 / \text{diag}(1, z) \rightarrow P_t \times_{U(2)} U(2) / \text{diag}(1, z) \rightarrow P_t \times_{U(2)} U(2) / T^2 \simeq P(E)$$

is the canonical line bundle over  $P(E)$ . The projection onto the first coordinate induces an isomorphism  $P_t \times_{U(2)} U(2) / \text{diag}(1, z) \simeq P_t / \text{diag}(1, z)$ , with circle action given by right multiplication with  $\text{diag}(e^{i\theta}, 1)$ . But this is the opposite orientation to the oriented circle

action on  $P_t/S_{0,1}^1 \rightarrow P_t/T^2$ . Thus this bundle is dual to the canonical line bundle, which proves our claim.

To see that  $c_1(P_t/S_{1,-1}^1) = x = \pi^*(x)$ , consider the diagram of circle fibrations:

$$\begin{array}{ccccc} U(2)/SU(2) & \longrightarrow & (P_t) \times_{SU(2)} \mathbb{C}P^1 & \longrightarrow & (P_t) \times_{U(2)} \mathbb{C}P^1 \simeq N_t \\ \downarrow & & \downarrow \pi_1 & & \downarrow \pi_1 = \pi \\ U(2)/SU(2) & \longrightarrow & P_t/SU(2) & \longrightarrow & P_t/U(2) \simeq \mathbb{C}P^2 \end{array}$$

The fibers of these bundles are oriented via the isomorphism induced by the homomorphism  $\det : U(2) \rightarrow S^1$  and hence right multiplication by  $\text{diag}(e^{i\theta}, 1) \subset U(2)$  induces a circle action with the correct orientation. The lower circle bundle is the Hopf bundle since by (1.7) it has Euler class  $c_1(P_t) = x$ . Thus  $c_1 = x$  for the upper circle bundle as well. But the total space is identified with  $(P_t) \times_{SU(2)} \mathbb{C}P^1 \simeq (P_t) \times_{SU(2)} SU(2) / \text{diag}(z, \bar{z}) \simeq P_t / \text{diag}(z, \bar{z}) \simeq P_t / S_{1,-1}^1$ . The circle action by  $U(2)/SU(2)$  is the right action by  $\text{diag}(e^{i\theta}, 1)$  on  $P_t/S_{1,-1}^1$ , whereas the natural circle action is given by right multiplication with  $\text{diag}(e^{i\psi}, e^{i\psi})$ . To see that both circle actions agree, observe that  $\text{diag}(e^{i\theta}, 1) = \text{diag}(e^{i\psi}, e^{i\psi}) \cdot \text{diag}(z, \bar{z})$  for  $z = e^{i\psi}$ ,  $\theta = 2\psi$  and that the action by  $\text{diag}(e^{i\psi}, e^{i\psi})$  is  $\mathbb{Z}_2$  ineffective.  $\square$

*Remark.* There is another natural basis  $\bar{x}, \bar{y}$  of  $H^2(N_t, \mathbb{Z})$  given by transgression in the fiber bundle  $T^2 \rightarrow P_t \rightarrow N_t$  of the natural basis in  $H^1(T^2, \mathbb{Z})$  corresponding to the splitting  $T^2 = \text{diag}(e^{i\theta}, e^{i\psi}) \subset U(2)$ . The Euler class of the circle bundle  $T^2/S_{a,b}^1 \rightarrow P_t/S_{a,b}^1 \rightarrow P_t/T^2$  is then given by  $-b\bar{x} + a\bar{y}$ . Thus Proposition 5.1 implies that  $\bar{x} = -y$ ,  $\bar{y} = x + y$  and hence  $e(M_{a,b}^t) = a\bar{x} + b\bar{y}$ .

In [GZ2] it was shown that  $U(2)$  principal bundles over  $\mathbb{C}P^2$  with  $w_2 \neq 0$  admit a metric with non-negative sectional curvature invariant under the action of  $U(2)$ . Hence, as a consequence of Proposition 5.1 and O’Neil’s formula we obtain:

**COROLLARY 5.2.** *The manifolds  $M_{a,b}^t$  admit a metric with non-negative sectional curvature for any integers  $a, b, t$ .*

Since  $S_{a,b}^1 \subset U(2)$  is conjugate to  $S_{b,a}^1 \subset U(2)$ , we have

**COROLLARY 5.3.** *The manifolds  $M_{a,b}^t$  and  $M_{b,a}^t$  are orientation preserving diffeomorphic.*

#### *Relationship with previously defined manifolds.*

**PROPOSITION 5.4.** *For  $t = \pm 1$  one has the following identifications.  $N_1$  is the homogeneous flag manifold  $SU(3)/T^2$  and  $M_{a,b}^1$  the Aloff-Wallach space  $W_{a,b}$ .  $N_{-1}$  is the inhomogeneous flag manifold  $SU(3)//T^2$  and  $M_{a,b}^{-1}$  is the Eschenburg space  $F_{a,b}$ .*

*Proof.* If we start with the homogeneous  $U(2)$  principal bundle  $U(2) \rightarrow SU(3) \rightarrow \mathbb{C}P^2$ , the associated 2-sphere bundle is  $SU(3)/T^2 \rightarrow \mathbb{C}P^2$  and  $SU(3)/Z \rightarrow \mathbb{C}P^2$  is its  $SO(3)$  principal

bundle. But  $SU(3)/Z \simeq SU(3)/\text{diag}(z, z, \bar{z}^2) \simeq W_{1,1}$ . Since  $W_{1,1}$  is simply connected and  $H^4(W_{1,1}, \mathbb{Z}) \cong \mathbb{Z}_3$ , (1.8) and (1.11) imply that the  $SO(3)$  bundle has  $p_1 = -3$  and  $w_2 \neq 0$ . Thus  $SU(3)/Z = Q_1$  and hence  $SU(3)/T^2 = N_1$ .

Similarly, there is a second free biquotient action of  $U(2)$  on  $SU(3)$ , see [E2], given by

$$B \star A = \text{diag}(1, 1, \det B^2) A \text{diag}(B, \det B)^{-1} \text{ where } B \in U(2), A \in SU(3).$$

By dividing by  $SU(2)$  first, one easily sees that  $SU(3)//U(2) \simeq \mathbb{C}\mathbb{P}^2$ , i.e. we obtain a  $U(2)$  principal bundle over  $\mathbb{C}\mathbb{P}^2$ . Now  $SU(3)//Z = \text{diag}(1, 1, z^4) \backslash SU(3)/\text{diag}(z, z, z^2)^{-1} = F_{1,1}$ . Since  $F_{1,1}$  is simply connected and  $H^4(F_{1,1}, \mathbb{Z}) \cong \mathbb{Z}_5$ , (1.8) and (1.11) imply that the  $SO(3)$  bundle has  $p_1 = 5$ . Thus  $SU(3)/Z = Q_{-1}$  and hence  $SU(3)/T^2 = N_{-1}$ .

Thus in both cases  $P_{\pm} = SU(3)$ , but with different actions by  $U(2)$ . Dividing by  $S_{a,b}^1$ , the last claim follows from Proposition 5.1.  $\square$

*Remark.* The diffeomorphism classification of  $M_{a,b}^1$  was carried out in [KS2]. Their choice of parameters  $a, b$  is the same as ours, but their orientation is opposite. Notice though that in the choice of  $m, n$ , one needs to change the sign of  $n$ , i.e.  $am + bn = 1$  in the formulas in Proposition 4.4.

The diffeomorphism classification of  $M_{a,b}^{-1}$  was carried out in [AMP1]. The correspondence of our and their parameters is  $a = l$ ,  $b = m$ ,  $m = a$ ,  $n = b$ , and their orientation is opposite. Notice though that the invariants of their Examples 7-9 in the Table on page 47 are incorrect.

The case of  $t = 0$  is also special:

**PROPOSITION 5.5.** *For  $t = 0$  we have  $N_0 = \mathbb{C}\mathbb{P}^3 \# \overline{\mathbb{C}\mathbb{P}^3}$  and  $P_0 = \mathbb{S}^5 \times \mathbb{S}^3$ . Furthermore,  $M_{a,b}^0 = \mathbb{S}^5 \times \mathbb{S}^3 / S^1$  with circle action  $(p, q) \rightarrow (z^{a+b}p, \text{diag}(z^a, z^b)q)$  where  $p \in \mathbb{C}^3, q \in \mathbb{C}^2$ .*

*Proof.* The bundle  $P_0$  has  $c_1 = x$  and  $c_2 = 0$  and its structure group thus reduces to  $U(1)$ . This reduced circle bundle must be the Hopf bundle since its Euler class is  $x$ . Hence  $N_0 = \mathbb{S}^5 \times_{S^1} \mathbb{S}^2$  which is well known to be diffeomorphic to  $\mathbb{C}\mathbb{P}^3 \# \overline{\mathbb{C}\mathbb{P}^3}$ .

Furthermore,  $P_0 = \mathbb{S}^5 \times_{U(1)} U(2)$  where  $U(1)$  acts on  $\mathbb{S}^5$  as the Hopf action and on  $U(2)$  by left multiplication with  $\text{diag}(z, 1)$ . One easily shows that the map  $\mathbb{S}^5 \times \mathbb{S}^3 = \mathbb{S}^5 \times SU(2) \rightarrow \mathbb{S}^5 \times_{U(1)} U(2) = P_0$  given by  $(p, A) \rightarrow [(p, A)]$  is a diffeomorphism. This easily implies that the action of  $U(2)$  on  $P_0$ , translated to  $\mathbb{S}^5 \times \mathbb{S}^3$ , is given by  $B \star (p, A) \rightarrow (\det B p, \text{diag}(\det \bar{B}, 1) A B)$  where  $p \in \mathbb{S}^5 \subset \mathbb{C}^3, A \in SU(2) \simeq \mathbb{S}^3$  and  $B \in U(2)$ . If we identify  $SU(2)$  with  $\mathbb{S}^3$  via its first column vector, we see that the circle action of  $B = \text{diag}(z^a, z^b) \in S_{a,b}^1 \subset U(2)$  on  $P_0 = \mathbb{S}^5 \times \mathbb{S}^3$  sends  $(p, (u, v))$  to  $(z^{a+b}p, (\bar{z}^b u, z^a v))$ , where  $(u, v) \in \mathbb{S}^3 \subset \mathbb{C}^2$ . But the circle action  $(u, v) \rightarrow (\bar{z}^b u, z^a v)$  on  $\mathbb{S}^3$  is equivalent to  $(u, v) \rightarrow (z^a u, z^b v)$  via conjugation on the first coordinate and a coordinate interchange. Since  $M_{a,b}^0 = P_0 / S_{a,b}^1$ , we obtain the last claim.  $\square$

*Remark.* Thus  $M_{a,b}^0$  are special examples of the 5-parameter family of manifolds mentioned at the beginning of Section 6. Together with Corollary 6.9 this implies that  $M_{1,1}^0 \simeq L_{-1,2} \simeq \bar{M}_{-1,2}$ .

We can regard the principal bundle  $Q_t$  as  $P_t/Z$  where  $Z = S_{1,1}^1$  is the center of  $U(2)$  and thus Proposition 5.1 implies

**COROLLARY 5.6.** *The  $SO(3)$  principal bundle  $Q_t$  with  $p_1(Q_t) = 1 - 4t$ ,  $w_2(Q_t) \neq 0$  is equal to  $M_{1,1}^t$ . Furthermore,  $Q_1 = W_{1,1}$  and  $Q_{-1} = F_{1,1}$ .*

*Natural diffeomorphisms between sphere and circle bundles.*

From the inclusions  $\text{diag}(z^p, z^q) \subset U(2)$  we obtain the fibration:

$$\mathbb{S}^3/\mathbb{Z}_{p+q} \simeq U(2)/\text{diag}(z^p, z^q) \rightarrow P_t/\text{diag}(z^p, z^q) \simeq M_{p,q}^t \rightarrow P_t/U(2) \simeq \mathbb{C}\mathbb{P}^2,$$

where the fiber is  $U(2)/\text{diag}(z^p, z^q) = SU(2)/\text{diag}(z^p, z^q)$  with  $z^{p+q} = 1$ . Hence the fiber is a lens space  $\mathbb{S}^3/\mathbb{Z}_{p+q}$ , and, in the case of  $p+q = \pm 1$ , we obtain a bundle with fiber  $\mathbb{S}^3$ . We can assume that  $p+q = 1$ , since replacing  $z$  by  $\bar{z}$  changes the sign of both  $p$  and  $q$ . Thus we obtain sphere bundles:

$$(5.7) \quad \mathbb{S}^3 \rightarrow M_{p,1-p}^t \rightarrow \mathbb{C}\mathbb{P}^2.$$

We now identify which sphere bundle this is.

**PROPOSITION 5.8.**  *$M_{p,1-p}^t$  is orientation preserving diffeomorphic to  $S_{-t,p(p-1)}$ .*

*Proof.* We compute the characteristic classes of the  $SO(4)$  principal bundle  $P^*$  over  $\mathbb{C}\mathbb{P}^2$  associated to the sphere bundle (5.7). For this we identify the induced  $SO(3)$  principal bundles  $P_{\pm}^* = P^*/S_{\pm}^3$  as discussed in Section 1. Notice though that the role of  $P$  and  $P^*$  are interchanged here.

Let  $\rho_p: U(2) \rightarrow U(2)$  be the homomorphism  $\rho_p(A) = (\det A)^{-p}A$  which we can also regard as a representation of  $U(2)$  on  $\mathbb{C}^2$ . The induced action of  $U(2)$  on the unit sphere  $\mathbb{S}^3(1) \subset \mathbb{R}^4 \cong \mathbb{C}^2$  is transitive with isotropy group at  $(1, 0)$  given by  $S_p^1 := \text{diag}(z^p, z^{1-p})$ , i.e.,  $\mathbb{S}^3(1) = U(2)/S_p^1$ . Consider now the associated vector bundle  $E_p = P_t \times_{U(2)} \mathbb{C}^2$  where  $U(2)$  acts on  $\mathbb{C}^2$  via  $\rho_p$  and  $E_p^{\mathbb{R}} = P \times_{U(2)} \mathbb{R}^4$  the underlying real bundle. For the sphere bundle of  $E_p^{\mathbb{R}}$  we have  $S(E_p^{\mathbb{R}}) = P \times_{U(2)} \mathbb{S}^3 = P \times_{U(2)} U(2)/S_p^1 = P/S_p^1$  and thus, using Proposition 5.1,  $S(E_p^{\mathbb{R}}) = M_{p,1-p}^t$ . We denote by  $P^*$  the  $SO(4)$  principal bundle associated to  $E_p^{\mathbb{R}}$ , i.e.  $P^* = P \times_{U(2)} SO(4)$ .

In order to view  $P^*$  in a different way, consider the following commutative diagram of homomorphisms:

$$\begin{array}{ccccc} \mathbb{S}^1 \times \mathbb{S}^3 & \xrightarrow{\tilde{\rho}_p} & \mathbb{S}^1 \times \mathbb{S}^3 & \xrightarrow{\tilde{\sigma}} & \mathbb{S}^3 \times \mathbb{S}^3 \\ \pi_1 \downarrow & & \pi_1 \downarrow & & \downarrow \pi_2 \\ U(2) & \xrightarrow{\rho_p} & U(2) & \xrightarrow{\sigma} & SO(4) \end{array}$$

where  $\pi_i$  are the two fold covers and  $\sigma, \tilde{\sigma}$  the embeddings discussed in Section 1. To make these diagrams commutative, one needs  $\tilde{\rho}_p(z, q) = (z^{-2p+1}, q) \in \mathbb{S}^1 \times \mathbb{S}^3$ .

Thus  $P^* = P \times_{U(2)} \mathrm{SO}(4) = P \times_{(S^1 \times S^3)/\Gamma} (S^3 \times S^3)/\Gamma$  and  $(z, q) \in S^1 \times S^3$  acts on  $(q_1, q_2) \in S^3 \times S^3$  as  $(z^{-2p+1}q_1, qq_2)$ . According to (1.6)

$$P_-^* = P/Z = Q_t$$

and thus  $p_1(P_-^*) = p_1(Q_t) = (1 - 4t)$ . On the other hand, by (1.6)

$$P_+^* = (P/\mathrm{SU}(2)) \times_{\mathrm{SO}(2)} \mathrm{SO}(3) = (P/\mathrm{SU}(2)) \times_{(S^1/\Gamma) \times \{e\}} [(S^3/\Gamma) \times \{e\}].$$

but  $S^1$  acts on  $S^3$  via  $z \star q = z^{-2p+1}q$ . The structure group of this bundle reduces to  $S^1$  with  $S^1$  principal bundle

$$(5.9) \quad S^1 \rightarrow (P/\mathrm{SU}(2)) \times_{(S^1/\Gamma) \times \{e\}} [(S^1/\Gamma) \times \{e\}] = (P/\mathrm{SU}(2))/\mathbb{Z}_{2p-1} \rightarrow P/U(2)$$

since the action of  $S^1$  on  $S^1$  has isotropy  $\mathbb{Z}_{2p-1}$ . Recall that by (1.7) the circle bundle  $P/\mathrm{SU}(2) \rightarrow P/U(2)$  has Euler class  $c_1(P) = x$ . The Gysin sequence then implies that the circle bundle (5.9) has Euler class  $e = \pm(2p-1)x$  and hence  $p_1(P_+^*) = e^2 = (1-2p)^2$ . By definition,  $p_1(P_-^*) = 4a+1$  and  $p_1(P_+^*) = 4b+1$ , and hence  $a = -t$  and  $b = p(p-1)$ , i.e.,  $M_{p,1-p}^t = S_{-t,p(p-1)}$ .

In order to see that the diffeomorphism is orientation preserving, we use Corollary 3.4 and Corollary 4.5 to see that the linking forms are the same.  $\square$

In particular:

**COROLLARY 5.10.** *For each integer  $p$ , there exist natural diffeomorphisms  $W_{p,1-p} \simeq S_{-1,p(p-1)}$  and  $F_{p,1-p} \simeq S_{1,p(p-1)}$ .*

*Remarks.* (a) Using the normalization of the Aloff-Wallach spaces described in Section 1, we see that  $W_{p,1} \simeq S_{-1,p(p+1)}$  with  $p \geq 0$  are the Aloff-Wallach spaces which are naturally  $S^3$  bundles over  $\mathbb{C}\mathbb{P}^2$ . They have positive curvature, unless  $p = 0$ . Using the diffeomorphism classification in Corollary 6.3, this gives rise to infinitely many diffeomorphisms between sphere bundles and  $W_{p,1}$ .

(b) The tangent bundle of  $\mathbb{C}\mathbb{P}^2$  is given by  $S_{-1,2} \simeq W_{2,-1} \simeq W_{1,1}$  since it has  $p_1 = e = 3$ . Thus  $W_{1,1}$  is both an  $S^3$  bundle over  $\mathbb{C}\mathbb{P}^2$ , as well as an  $\mathrm{SO}(3)$  principal bundle over  $\mathbb{C}\mathbb{P}^2$ .

(c) The Eschenburg spaces  $F_{p,1-p}$  do not have positive curvature in the Eschenburg metric since  $p(1-p) \leq 0$ .

## 6. SPIN $S^3$ AND $S^1$ BUNDLES OVER $\mathbb{C}\mathbb{P}^2$

In this section we discuss bundles which are spin. One class of such bundles has a total space which is spin and can hence be diffeomorphic to an Eschenburg space. Some of the other classes admit Einstein metrics and we will compare them as well.

There is a specific class of manifolds of type  $\bar{E}_r$  which has been considered previously in [WZ], where it was shown to admit Einstein metrics. These manifolds are total spaces of circle bundles  $S^1 \rightarrow L_{a,b} \rightarrow \mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^1$  with Euler class  $e = ax + by$  and  $\mathrm{gcd}(a,b) = 1$ , where  $x$  and  $y$  are the natural generators in the first or second factor. They can also be considered as the base space of a circle bundle:  $S^1 \rightarrow S^5 \times S^3 \rightarrow L_{a,b}$  where  $S^1$  acts on  $S^5 \times S^3 \subset \mathbb{C}^3 \oplus \mathbb{C}^2$  as  $(u, v) \rightarrow (z^{-b}u, z^av)$ . Thus  $L_{1,0} = S^5 \times \mathbb{C}\mathbb{P}^1$  and  $L_{0,1} = \mathbb{C}\mathbb{P}^2 \times S^3$ , and

by conjugating in each component, one sees that  $L_{a,b} = L_{\pm a, \pm b}$ . Furthermore,  $L_{a,b}$ ,  $a \neq 0$ , has cohomology type  $E_{a^2}$  if  $b$  is even and cohomology type  $\bar{E}_{a^2}$  if  $b$  is odd. Projecting to the first component, one obtains a lens space bundle  $\mathbb{S}^3/\mathbb{Z}_{|b|} \rightarrow L_{a,b} \rightarrow \mathbb{C}\mathbb{P}^2$ . Thus  $L_{a,1}$  is naturally an  $\mathbb{S}^3$  bundle over  $\mathbb{C}\mathbb{P}^2$ . See [WZ] for details.

These manifolds were later also discussed in [KS1], where they computed the Kreck-Stolz invariants and exhibited certain diffeomorphism among them, which gave rise to counter examples to a conjecture by W.Y.Hsiang.

In [Kr1] the manifolds  $L_{a,b}$  were generalized to a 5 parameter family of manifolds by dividing  $\mathbb{S}^5 \times \mathbb{S}^3$  by the  $\mathbb{S}^1$  action  $((u_1, u_2, u_3), (v_1, v_2)) \rightarrow ((z^{a_1}u_2, z^{a_2}u_3, z^{a_3}), (z^{b_1}v_1, z^{b_2}v_2))$ , and in [Es], their Kreck-Stolz invariants were computed if all  $a_i$  are equal. These manifolds have cohomology type  $E_r$  or  $\bar{E}_r$ , depending on whether  $\sum a_i + \sum b_i$  is even or odd, and  $r = b_1b_2$ .

**6.1. 3 sphere bundles which are spin.** Here we consider the bundles

$$\mathbb{S}^3 \rightarrow \bar{S}_{a,b} \rightarrow \mathbb{C}\mathbb{P}^2 \text{ with } p_1(\bar{S}_{a,b}) = 2a + 2b, \quad e(\bar{S}_{a,b}) = a - b \text{ and } w_2 = 0$$

Recall that, according to (1.10), these are the allowed values in the spin case. As in Section 3, one easily proves:

**PROPOSITION 6.1.** *If  $r = |a - b| \geq 1$  one has the following:*

- (a) *The manifolds  $\bar{S}_{a,b}$  have cohomology type  $\bar{E}_r$  and  $p_1(T\bar{S}_{a,b}) \equiv 2a + 2b + 3 \pmod{r}$ . Furthermore, the linking form is given by  $lk(\bar{S}_{a,b}) = \frac{1}{a-b}$ .*
- (b)  *$\bar{S}_{a,b}$  has non-negative curvature if  $a$  and  $b$  are both even.*

Part (b) follows from [GZ2]. It is not known whether  $\bar{S}_{a,b}$  admits a metric with non-negative curvature if  $a$  and  $b$  are not both even, although they do if  $a$  is even and  $b = (2r + 1)^2$ .

For the Kreck-Stolz invariants one obtains:

**PROPOSITION 6.2.** *The Kreck-Stolz invariants for  $\bar{S}_{a,b}$  with  $a \neq b$  are given by:*

$$\begin{aligned} s_1(\bar{S}_{a,b}) &\equiv \frac{1}{2^7 \cdot 7 \cdot (a-b)} (2a + 2b + 3)^2 - \frac{1}{2^7 \cdot 3 \cdot (a-b)} (4a + 4b + 5) - \frac{\text{sgn}(a-b)}{2^5 \cdot 7} \pmod{1} \\ s_2(\bar{S}_{a,b}) &\equiv \frac{-1}{2^2 \cdot 3 \cdot (a-b)} (a + b - 1) \pmod{1} \\ s_3(\bar{S}_{a,b}) &\equiv \frac{-1}{2^2 \cdot (a-b)} (a + b - 5) \pmod{1}. \end{aligned}$$

This easily implies the following homeomorphism and diffeomorphism classification:

**COROLLARY 6.3.** *The manifolds  $\bar{S}_{a,b}$  and  $\bar{S}_{a',b'}$  with  $r = a - b = a' - b' > 0$  are*

- (a) *orientation preserving homeomorphic if and only if  $a \equiv a' \pmod{6r}$ .*
- (b) *orientation preserving diffeomorphic if and only if*

$$a \equiv a' \pmod{6r} \text{ and } (a - a') [3(a + a') - 3r + 1] \equiv 0 \pmod{2^3 \cdot 3 \cdot 7 \cdot r}.$$

- (a') *orientation reversing homeomorphic if and only if  $r = 1$  and  $a + a' \equiv 2 \pmod{6}$  or  $r = 2$  and  $a + a' \equiv 3 \pmod{12}$ .*

(b') *orientation reversing diffeomorphic if and only if*

$$a + a' \equiv 2 \pmod{6} \quad \text{and} \quad a(3a - 2) \equiv -a'(3a' - 2) + 2 \pmod{2^3 \cdot 3 \cdot 7} \quad \text{for } r = 1$$

*or*

$$a + a' \equiv 3 \pmod{12} \quad \text{and} \quad a(3a - 5) \equiv -a'(3a' - 5) \pmod{2^4 \cdot 3 \cdot 7} \quad \text{for } r = 2.$$

**6.2. Circle bundles over 2 sphere bundles which are spin.** In this section we discuss the manifolds  $\bar{M}_{a,b}^t$  arising from  $\mathbb{S}^2$  bundles over  $\mathbb{C}\mathbb{P}^2$  which are spin. According to (1.8), the corresponding  $\text{SO}(3)$  principal bundle satisfies  $p_1 \equiv 0 \pmod{4}$  and we define:

$$\mathbb{S}^2 \rightarrow \bar{N}_t \xrightarrow{\pi} \mathbb{C}\mathbb{P}^2 \quad \text{with} \quad p_1(\bar{N}_t) = 4tx^2 \quad \text{and} \quad w_2 = 0$$

for some integer  $t$  and we denote the corresponding  $\text{SO}(3)$  principal bundle by  $\bar{Q}_t$ . If we define the  $\text{U}(2)$  principal bundle

$$\text{U}(2) \rightarrow \bar{P}_t \rightarrow \mathbb{C}\mathbb{P}^2 \quad \text{with} \quad c_1(\bar{P}_t) = 0 \quad \text{and} \quad c_2(\bar{P}_t) = -tx^2$$

we again have that

$$\bar{N}_t \simeq P(\bar{E}) \simeq \bar{P}_t / \mathbb{T}^2 \quad \text{and} \quad \bar{P}_t / Z = \bar{Q}_t$$

where  $\bar{E} = \bar{P}_t \times_{\text{U}(2)} \mathbb{C}^2$ . For the cohomology ring of  $\bar{N}_t$  we thus obtain

$$(6.4) \quad \begin{aligned} H^2(\bar{N}_t) &\cong \mathbb{Z} \oplus \mathbb{Z} \text{ with generators } x, y; \\ H^4(\bar{N}_t) &\cong \mathbb{Z} \oplus \mathbb{Z} \text{ with generators } x^2, xy \text{ and relationship } y^2 = tx^2; \\ H^6(\bar{N}_t) &\cong \mathbb{Z} \text{ with generator } x^2y \text{ and } x^3 = 0, y^2x = 0, y^3 = tx^2y. \end{aligned}$$

where  $y = c_1(S^*)$  and  $S^*$  is the dual of the tautological complex line bundle  $S$  over  $P(\bar{E})$ .

We now define the circle bundles over  $\bar{N}_t$  via

$$\mathbb{S}^1 \rightarrow \bar{M}_{a,b}^t \xrightarrow{\sigma} \bar{N}_t \quad \text{with} \quad e(\bar{M}_{a,b}^t) = ax + by$$

and  $(a, b) = 1$ . One of the differences with  $N_t$  is that  $\bar{N}_t$  is not spin and hence  $\bar{M}_{a,b}^t$  may or may not be spin. One easily sees

**PROPOSITION 6.5.** *If  $r = |a^2 - tb^2|$  with  $r \neq 0$ , the manifolds  $\bar{M}_{a,b}^t$  have cohomology type  $\bar{E}_r$  if  $b$  is odd, and cohomology type  $E_r$  if  $b$  is even. Furthermore, the first Pontryagin class is  $p_1(T\bar{M}_{a,b}^t) \equiv (3 + 4t)b^2 \pmod{r}$ .*

*Remark.* The homotopy invariant  $\pi_4(\bar{M}_{a,b}^{2t}) = \pi_4(\bar{N}_t)$  again does not depend on  $a, b$ . We suspect that, as in the case of non-spin bundles,  $\pi_4(\bar{M}_{a,b}^t) = 0$  if  $t$  odd, and  $\pi_4(\bar{M}_{a,b}^t) = \mathbb{Z}_2$  if  $t$  even. For example, it follows from Corollary 6.9 that  $\pi_4(\bar{M}_{a,b}^0) = \mathbb{Z}_2$ .

In order to describe the circle bundles in a different way, as in Proposition 5.1, we consider  $\bar{N}_t$  as the projectivization of the vector bundle  $\bar{E} = \bar{P}_t^* \times_{\text{U}(2)} \mathbb{C}^2$  where  $c_1(\bar{P}_t^*) = 2x$  and  $c_2(\bar{P}_t^*) = -(t-1)x^2$ . Since  $c_1^2 - 4c_2 = 4tx^2$  and since  $w_2 \equiv c_1 \pmod{2} = 0$ , the projectivization of the vector bundle associated to  $\bar{P}_t^*$  is again  $\bar{N}_t$  and  $\bar{P}_t^* / \mathbb{T}^2 = \bar{N}_t$  as well. But notice that  $\pi_1(\bar{P}_t^*) \cong \mathbb{Z}_2$  since in the spectral sequence for the  $\text{U}(2)$  principal bundle  $d_2$  takes a generator in  $H^1(\text{U}(2), \mathbb{Z}) \cong \mathbb{Z}$  to  $c_1 = 2x$ . Similarly, since  $w_2(\bar{Q}_t) = 0$ , (1.11)



implies that  $\pi_1(\bar{Q}_t) \cong \mathbb{Z}_2$ . We let  $\bar{P}'_t$  and  $\bar{Q}'_t$  be the universal covers of  $\bar{P}_t^*$  and  $\bar{Q}_t$ . We thus obtain the spin bundles

$$S^1 \times S^3 \rightarrow \bar{P}'_t \rightarrow \mathbb{C}\mathbb{P}^2 \quad , \quad S^3 \rightarrow \bar{Q}'_t \rightarrow \mathbb{C}\mathbb{P}^2 \quad \text{with} \quad \bar{P}'_t / \mathbb{T}^2 = \bar{N}_t = \bar{Q}'_t / \text{SO}(2).$$

We can now formulate the analogue to Proposition 5.1:

**PROPOSITION 6.6.** *The circle bundle  $S^1 \rightarrow \bar{M}_{a,b}^t \rightarrow \bar{N}_t$  can be equivalently described as the circle bundle  $\mathbb{T}^2 / S_{-b,a}^1 \rightarrow \bar{P}'_t / S_{-b,a}^1 \rightarrow \bar{P}'_t / \mathbb{T}^2$ , where  $S_{a,b}^1 = \text{diag}(z^a, z^b) \subset S^1 \times S^1 \subset S^1 \times S^3$ .*

*Proof.* The proof is similar to the proof of Proposition 5.1 and we indicate the changes that need to be made as one goes from  $U(2)$  principal bundles to  $S^1 \times S^3$  principal bundles.

In the proof of Proposition 5.1 we showed that if  $P \rightarrow \mathbb{C}\mathbb{P}^2$  is a  $U(2)$  principal bundle, the circle bundle  $P / S_{r,s}^1 \rightarrow P / \mathbb{T}^2$  has Euler class linear in  $r$  and  $s$  and  $e(P / S_{0,1}^1) = c_1(S^*)$  and  $e(P / S_{1,-1}^1) = c_1(P)$ . These results did not depend on whether the bundle is spin or not. We now apply this to the bundle  $\bar{P}^*$  above. Notice though that if we describe  $\bar{N}_t$  as  $\bar{P}'_t / \mathbb{T}^2$ , the cohomology ring is expressed in terms of a different basis  $x, y'$  and now  $y'^2 = -2xy' + (t-1)x^2$ , whereas in the  $x, y$  basis we have  $y^2 = tx^2$ . On the other hand, the only elements  $z \in H^2(\bar{N}_t)$  with  $z^2 = tx^2$  are a multiple of  $x$ , or  $z = \pm y$ . But if  $z = \alpha x + \beta y'$ , we have  $z^2 = tx^2$  only if  $\beta = 0$ , or  $\alpha = \beta = \pm 1$ . Since  $y$  is not a multiple of  $x$ , this implies that  $z = \pm(x + y')$ . Thus, depending on a sign  $\epsilon = \pm 1$ ,  $y = \epsilon(x + y')$ . As indicated above,  $e(\bar{P}'_t / S_{0,1}^1) = y'$  and  $e(\bar{P}'_t / S_{1,-1}^1) = c_1(\bar{P}'_t) = 2x$ . This implies that  $e(\bar{P}'_t / S_{r,s}^1) = 2rx + (r+s)y' = (r-s)x + \epsilon(r+s)y$  and hence  $\bar{P}'_t / S_{r,s}^1 = \bar{M}_{(r-s, \epsilon(r+s))}^t$ .

Recall that in the two fold cover  $S^1 \times S^3 \rightarrow U(2)$  the circle  $S_{a,b}^1$  is mapped to the circle  $S_{a+b, a-b}^1$  and thus  $\bar{P}' / S_{a,b}^1 \rightarrow \bar{P}' / S_{a+b, a-b}^1 = \bar{M}_{2b, \epsilon 2a}^t$  is a two fold cover. But from the Gysin sequence it follows that  $\pi_1(M_{2b, \epsilon 2a}^t) = \mathbb{Z}_2$  and thus  $M_{b, \epsilon a}^t$  is the only two fold cover and hence  $\bar{P}' / S_{a,b}^1 = M_{b, \epsilon a}^t$ . In other words,  $M_{a,b}^t$  is orientation preserving diffeomorphic to  $\bar{P}' / S_{\epsilon b, a}^1$ .

We now claim that we can make an arbitrary choice in the value of  $\epsilon$ . Indeed, by conjugating with  $(1, j) \in S^1 \times S^3$ , we see that the circles  $S_{a,b}^1$  and  $S_{a,-b}^1$  are conjugate in  $S^1 \times S^3$ . But notice that this conjugation also reverses the orientation of the circle. Changing the sign of  $a$  and  $b$  changes the sign of the Euler class, and hence the orientation. Thus  $\bar{P}' / S_{a,b}^1$  and  $\bar{P}' / S_{-a,b}^1$  are orientation preserving diffeomorphic. We will make the choice of  $\epsilon = -1$  for symmetry reasons.  $\square$

*Remark.* In this description of  $\bar{M}_{a,b}^t$  it is important that we choose the bundle  $\bar{P}'_t$  instead of either  $\bar{P}_t$  or  $\bar{P}_t^*$ . Indeed, following the same proof, one sees that  $e(\bar{P}_t / S_{1,-1}^1) = 0$  and hence  $e(\bar{P}_t / S_{r,s}^1) = (r+s)y$ , i.e. among the circle bundles  $\bar{P}_t / S_{r,s}^1 \rightarrow \bar{N}_t$  we obtain up to covers only one bundle. In terms of the circle bundles  $\bar{P}'_t / S_{r,s}^1 \rightarrow \bar{N}_t$ , the proof also shows that  $\bar{P}'_t / S_{r,s}^1 = \bar{M}_{r-s, -(r+s)}^t$ , which only gives half of the manifolds  $\bar{M}_{a,b}^t$ .

We can now discuss which of these manifolds carry a metric with non-negative curvature:

**COROLLARY 6.7.** *The manifolds  $\bar{M}_{a,b}^t$  with  $t$  even admit a metric with non-negative curvature.*

*Proof.* In [GZ2] it was shown that any  $U(2)$  principal bundle with  $w_2 = 0$  and  $c_1^2 - 4c_2$  divisible by 8 admits a metric with nonnegative curvature invariant under  $U(2)$ . Since for  $\bar{P}_t^*$  we have  $c_1^2 - 4c_2 = 4t$ , it admits such a metric when  $t$  is even. Thus the two fold cover  $\bar{P}_t'$  admits a non-negatively curved metric invariant under  $S^1 \times S^3$  and since by Proposition 6.6 we have  $\bar{P}_t'/S_{-b,a}^1 = \bar{M}_{a,b}^t$ , and O'Neil's formula implies the claim.  $\square$

We now discuss various natural diffeomorphisms.

**COROLLARY 6.8.** *The manifold  $\bar{M}_{a,b}^t$  is orientation preserving diffeomorphic to  $\bar{M}_{-a,b}^t$ , and orientation reversing diffeomorphic to  $\bar{M}_{a,-b}^t$  and  $\bar{M}_{-a,-b}^t$ . Thus  $\bar{M}_{\pm a, \pm b}^t$  are all diffeomorphic to each other.*

This follows from Proposition 6.6, together with the observation that  $\bar{P}'/S_{a,b}^1$  and  $\bar{P}'/S_{-a,b}^1$  are orientation preserving diffeomorphic, see the end of the proof of Proposition 6.6.

Next, recall that we have circle bundles  $L_{a,b} \rightarrow \mathbb{C}P^2 \times \mathbb{C}P^1$  with Euler class  $e = ax + by$ .

**COROLLARY 6.9.** *The manifold  $\bar{M}_{a,b}^0$  is naturally diffeomorphic to  $L_{a,b}$ .*

*Proof.* The bundle  $\bar{N}_t$  with  $t = 0$  is trivial since  $p_1 = 0$  and  $w_2 = 0$ . Thus  $\bar{N}_0 = \mathbb{C}P^2 \times \mathbb{C}P^1$ . In order to identify  $\bar{P}'_0$ , we start with  $R = S^5 \times S^3$  and define a free action by  $S^1 \times S^3$  on  $R$  as follows.  $S^1 \subset S^1 \times S^3$  acts as the Hopf action on the first factor and  $S^3 \subset S^1 \times S^3$  acts via left multiplication on the second factor  $S^3$ , regarded as the Lie group  $Sp(1)$ . Since  $R/S^1 \times S^3 = \mathbb{C}P^2$  we thus have a principal bundle  $S^1 \times S^3 \rightarrow R \rightarrow \mathbb{C}P^2$  and we claim that this principal bundle is  $\bar{P}'_0$ . To see this, we apply (1.6) to the  $U(2)$  principal bundle  $R^* = R/\mathbb{Z}_2$  with  $\mathbb{Z}_2$  generated by  $(-1, -1) \in S^1 \times S^3$ . Clearly,  $R^*/SU(2) = \mathbb{R}P^5$  and hence the circle bundle (1.7) has Euler class  $2x$  which implies  $c_1(R^*) = 2x$ . Furthermore,  $P_- = R^*/Z = R/[S^1 \times \{\pm 1\}] = \mathbb{C}P^2 \times SO(3)$ . Thus  $0 = p_1(P_-) = c_1^2 - 4c_2$  which implies  $c_2(R^*) = x^2$ . Thus  $R^* = \bar{P}'_0$  and hence  $\bar{P}'_0 = R = S^5 \times S^3$ . Now Proposition 6.6 implies our claim.  $\square$

There are again natural diffeomorphisms between the circle bundles and 3-sphere bundles as in (5.8). From the inclusion  $S_{a,b}^1 = \text{diag}(z^a, z^b) \subset S^1 \times S^3$  we obtain lens space bundles:

$$\mathbb{S}^3/\mathbb{Z}_{|b|} = S^1 \times S^3 / \text{diag}(z^{-b}, z^a) \rightarrow \bar{P}'_t / \text{diag}(z^{-b}, z^a) = \bar{M}_{a,b}^t \rightarrow P_t / S^1 \times S^3 = \mathbb{C}P^2,$$

which are 3-sphere bundles if  $b = \pm 1$ , and we can assume that  $b = 1$ .

**PROPOSITION 6.10.** *The manifold  $\bar{M}_{k,1}^t$  is orientation preserving diffeomorphic to  $\bar{S}_{t,k^2}$ .*

*Proof.* The proof is similar to Proposition 5.8 and we indicate the changes one needs to make. Let  $\rho_k$  be the representation of  $S^1 \times S^3$  on  $\mathbb{R}^4 \simeq \mathbb{H}$  given by  $v \in \mathbb{H} \rightarrow z^k v \bar{q}$ ,  $(z, q) \in S^1 \times S^3$ . We then have the associated vector bundle  $\bar{P}'_t \times_{S^1 \times S^3} \mathbb{R}^4$ . The action on the sphere  $S^3 \subset \mathbb{R}^4$  is transitive with isotropy  $(z, z^k)$  and hence the sphere bundle of this vector bundle is diffeomorphic to  $\bar{P}'_t / S_{1,k}^1 = \bar{M}_{k,-1}^t$ .

As in the proof of Proposition 5.8, we let  $\hat{P}_t = \bar{P}'_t \times_{S^1 \times S^3} (S^3 \times S^3 / \Gamma)$  be the  $SO(4)$  principal bundle corresponding to this sphere bundle. Here  $(z^k, q) \in S^1 \times S^3$  acts on  $S^3 \times S^3$  as left multiplication. The remaining computations are similar:

$$\begin{aligned}\hat{P}_- &= [\bar{P}'_t \times_{S^1 \times S^3} (S^3 \times S^3)/\Gamma] / S^3 \times \{e\} = \bar{P}'_t \times_{S^1 \times S^3} [\{e\} \times (S^3/\Gamma)] \\ &= \hat{P}'_t / [S^1 \times \{\pm 1\}] = \bar{P}^*/Z = \bar{Q}_t^*\end{aligned}$$

and thus  $p_1(\hat{P}_-) = 4t$ . Furthermore,

$$\begin{aligned}\hat{P}_+ &= [\bar{P}'_t \times_{S^1 \times S^3} (S^3 \times S^3)/\Gamma] / \{e\} \times S^3 = \bar{P}'_t \times_{S^1 \times S^3} [(S^3/\Gamma) \times \{e\}] \\ &= (\bar{P}'/\{e\} \times S^3)_{S^1 \times \{e\}} [(S^3/\Gamma) \times \{e\}] = (\bar{P}^*/\text{SU}(2))_{(S^1/\Gamma) \times \{e\}} [(S^3/\Gamma) \times \{e\}]\end{aligned}$$

This bundle reduces to the circle bundle

$$(\bar{P}^*/\text{SU}(2))_{(S^1/\Gamma) \times \{e\}} [(S^1/\Gamma) \times \{e\}] = (\bar{P}^*/\text{SU}(2))/\mathbb{Z}_k$$

Since the  $S^1$  bundle  $\bar{P}^*/\text{SU}(2) \rightarrow \bar{P}^*/\text{U}(2)$  has Euler class  $c_1 = 2x$ , this reduced bundle has Euler class  $2kx$  and hence  $p_1(\hat{P}_+) = 4k^2$ . Thus  $a = t$  and  $b = k^2$ . To see that the diffeomorphism is orientation preserving we use Proposition 6.1 and Corollary 6.13 to see that the linking form is the same.  $\square$

Finally we observe:

**COROLLARY 6.11.** *The principal  $S^3$  bundle over  $\mathbb{C}\mathbb{P}^2$  with Euler class  $tx^2$  is orientation preserving diffeomorphic to  $\bar{M}_{0,1}^t \simeq \bar{S}_{t,0}$ .*

*Proof.* Recall from Section 1 that  $\bar{P}_t/Z = \bar{Q}_t$  where  $Z$  is the center of  $\text{U}(2)$ , and hence similarly  $\bar{P}'_t/[S^1 \times \{e\}] = \bar{Q}'_t$ . Hence Proposition 6.6 implies that  $\bar{Q}'_t = \bar{M}_{0,1}^t$ . By assumption, we have  $p_1(\bar{Q}_t) = 4tx^2$ . This implies that  $e(\bar{Q}'_t) = tx^2$  since the homomorphism  $S^3 \rightarrow \text{SO}(3)$  induces multiplication by 4 on  $H^4(B_{S^3}, \mathbb{Z}) \simeq \mathbb{Z} \rightarrow H^4(B_{\text{SO}(3)}, \mathbb{Z}) \simeq \mathbb{Z}$  and the generator in the first group is the Euler class and in the second group the Pontryagin class.  $\square$

For the Kreck-Stolz invariants we have:

**PROPOSITION 6.12.** *If  $s = a^2 - b^2t \neq 0$ , the Kreck-Stolz invariants for  $\bar{M}_{a,b}^t$  are given by:*

*b even*

$$s_1(\bar{M}_{a,b}^t) \equiv -\frac{1}{2^5 \cdot 7} \text{sign}(W) + \frac{b}{2^7 \cdot 7} (6 + 8t + 3a^2 + b^2t) - \frac{b}{2^7 \cdot 7 \cdot s} (3 + 4t)^2 \pmod{1}$$

$$\begin{aligned}
s_2(\bar{M}_{a,b}^t) &\equiv -\frac{1}{2^4 \cdot 3} \{b(n^2 + tm^2) - 2anm\} \\
&\quad - \frac{1}{2^4 \cdot 3 \cdot s} \{4nm\alpha - [3 + 4t - 2(n^2 + tm^2)]\beta\} \pmod{1}, \\
s_3(\bar{M}_{a,b}^t) &\equiv -\frac{1}{2^2 \cdot 3} \{b(n^2 + tm^2) - 2anm\} \\
&\quad - \frac{1}{2^2 \cdot 3 \cdot s} \{16nm\alpha - [3 + 4t - 8(n^2 + tm^2)]\beta\} \pmod{1},
\end{aligned}$$

$b$  odd

$$\begin{aligned}
s_1(\bar{M}_{a,b}^t) &\equiv -\frac{1}{2^5 \cdot 7} \text{sign}(W) + \frac{b}{2^7 \cdot 7} (6 + 8t + 3a^2 + b^2t) - \frac{b}{2^7 \cdot 7 \cdot s} (3 + 4t)^2 \\
&\quad - \frac{1}{2^6 \cdot 3} \{b(n^2 + tm^2) - 2anm\} \pmod{1}, \\
&\quad + \frac{1}{2^7 \cdot 3 \cdot s} \{-2nm\alpha + (6 + 8t - n^2 - tm^2)\beta\} \\
s_2(\bar{M}_{a,b}^t) &\equiv -\frac{1}{2^3 \cdot 3} \{b(n^2 + tm^2) - 2anm\} \\
&\quad - \frac{1}{2^3 \cdot 3 \cdot s} \{10nm\alpha - [3 + 4t - 5(n^2 + tm^2)]\beta\} \pmod{1}, \\
s_3(\bar{M}_{a,b}^t) &\equiv -\frac{1}{2^3} \{b(n^2 + tm^2) - 2anm\} \\
&\quad - \frac{1}{2^3 \cdot s} \{26nm\alpha - [3 + 4t - 13(n^2 + tm^2)]\beta\} \pmod{1},
\end{aligned}$$

where

$$\alpha = a(n^2 + tm^2) + 2tbnm \quad , \quad \beta = b(n^2 + tm^2) + 2anm,$$

and  $n, m \in \mathbb{Z}$  are chosen such that  $am + bn = 1$ . Furthermore, if  $b$  is odd, we additionally require that  $m$  is odd as well. Also,

$$\text{sign}(W) = \begin{cases} 0, & \text{if } s > 0 \\ 2, & \text{if } s < 0 \text{ and } b(t+1) > 0 \\ -2, & \text{if } s < 0 \text{ and } b(t+1) < 0. \end{cases}$$

*Proof.* The proof is similar to Proposition 4.4, and we indicate the changes that are needed. A natural choice for a bounding manifold is the disk bundle  $\sigma': \bar{W}_{a,b}^8 \rightarrow \bar{N}_t$  of the rank 2 vector bundle  $E^2$  associated to the circle bundle  $\sigma$ . One easily shows that

$$p_1(T\bar{W}) = (3 + 4t)x^2 + e^2, \quad w_2(T\bar{W}) \equiv (1 + a)x + by \pmod{2}.$$

where  $e = ax + by$ . Following the proof of Proposition 4.3 there exists a generator  $u \in H^2(\bar{M}_{a,b}^t; \mathbb{Z}) \cong \mathbb{Z}$  with  $\sigma^*(x) = -bu$  and  $\sigma^*(y) = au$  and  $w_2(T\bar{M}_{a,b}^t) = bu \pmod{2}$ . Thus the manifolds  $\bar{M}_{a,b}^t$  and  $\bar{W}_{a,b}$  are both spin if  $b$  is even and both non-spin if  $b$  is odd. Hence we can choose  $c = 0$  in (2) and (2). Next we need to choose a class  $z \in H^2(W, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$  with  $z|_{\partial W} = u$ . For this we let  $m, n$  be integers with  $am + bn = 1$  and set  $z = -nx + my$ . It follows that  $\sigma^*(z) = u$  and thus  $z$  has the required properties in (2.1).

Note that in the case of  $b$  even, we obtain  $w_2(T\bar{W}) \equiv (1+a)x + by \pmod{2}$  as required. However, in the case of  $b$  odd, we have  $w_2(T\bar{W}) \equiv c + z \pmod{2} \equiv -nx + my \pmod{2}$ . For this to be equivalent to  $(1+a)x + by \pmod{2}$ , we additionally need to choose  $m$  to be odd (which one easily sees implies  $n \equiv 1+a \pmod{2}$  as required). Note that it is always possible to choose such an  $m$ .

Replacing the value of  $p_1$  in (2) and (2) we obtain

$b$  even

$$\begin{aligned} S_1(\bar{W}, z) &= -\frac{1}{2^5 \cdot 7} \text{sign}(W) + \frac{1}{2^7 \cdot 7} ((3+4t)^2 x^4 + 2(3+4t)x^2 e^2 + e^4), \\ S_2(\bar{W}, z) &= -\frac{1}{2^4 \cdot 3} ((3+4t)z^2 x^2 + z^2 e^2 - 2z^4), \\ S_3(\bar{W}, z) &= -\frac{1}{2^2 \cdot 3} ((3+4t)z^2 x^2 + z^2 e^2 - 8z^4). \end{aligned}$$

$b$  odd

$$\begin{aligned} S_1(\bar{W}, z) &= -\frac{1}{2^5 \cdot 7} \text{sign}(W) + \frac{1}{2^7 \cdot 3 \cdot 7} [3(3+4t)^2 x^4 + 6(3+4t)x^2 e^2 + 3e^4 \\ &\quad - 14(3+4t)z^2 x^2 - 14z^2 e^2 + 7z^4], \\ S_2(\bar{W}, z) &= -\frac{1}{2^3 \cdot 3} ((3+4t)z^2 x^2 + z^2 e^2 - 5z^4), \\ S_3(\bar{W}, z) &= -\frac{1}{2^3} ((3+4t)z^2 x^2 + z^2 e^2 - 13z^4). \end{aligned}$$

We choose an orientation for  $\bar{N}_t$  such that  $\langle x^2 y, [\bar{N}_t] \rangle = 1$ . The orientation for the vector bundle  $E^2$  defines a Thom class  $U \in H^2(W, \partial W) \cong \mathbb{Z}$  and we define the orientation on  $\bar{W}$  such that  $U \cap [\bar{W}, \partial \bar{W}] = [\bar{N}_t]$ . On  $\bar{M}_{a,b}^t = \partial \bar{W}$  we pick the orientation induced by the orientation on  $\bar{W}$ . Using  $j^*(U) = e(E^2) = e$  with  $j: \bar{W} \rightarrow (\bar{W}, \partial \bar{W})$ , one easily computes the characteristic numbers:

$$\begin{aligned} e^4 &= b(3a^2 + b^2 t) \\ e^2 z^2 &= -2anm + bn^2 + btm^2 \\ x^2 e^2 &= b \end{aligned}$$

For the characteristic numbers involving  $x$  and  $z$  note that  $sx^2 = (ax - by) \cup e$  and  $sz^2 = (\alpha x - \beta y) \cup e$ , where  $\alpha = atm^2 + an^2 + 2tbnm$  and  $\beta = btm^2 + bn^2 + 2anm$ . Thus

$$\begin{aligned} sz^4 &= -2nm\alpha - (n^2 + tm^2)\beta \\ sz^2 x^2 &= -\beta \\ sx^4 &= -b \end{aligned} .$$

One easily shows that the signature matrix is given by  $\begin{pmatrix} b & a \\ a & bt \end{pmatrix}$  and since  $\det R = -s$  and  $\text{tr } R = b(t+1)$ , we obtain  $\text{sign } W$  as claimed. Substituting into (2) and (2) finishes the proof.  $\square$

Recall that the linking form is equal to the characteristic number  $z^4$  and we hence obtain:

**COROLLARY 6.13.** *If  $s = a^2 - b^2 t$  and  $r = |s| > 1$ , the linking form of  $\bar{M}_{a,b}^t$  is given by*

$$lk(\bar{M}_{a,b}^t) = -\frac{1}{s} [bn^4 + 6bt n^2 m^2 + 4atn m^3 + 4an^3 m + bt^2 m^4] \in \mathbb{Q}/\mathbb{Z}$$

where  $m, n \in \mathbb{Z}$  are chosen such that  $am + bn = 1$ , and if  $b$  is odd,  $m$  is odd as well.

*Remark.* The diffeomorphism classification of the manifolds  $L_{a,b} = \bar{M}_{a,b}^0$  was carried out in [KS1]. For comparison, note that Proposition 6.6 implies that  $x, y \in H^2(\bar{N}_t, \mathbb{Z})$  are the transgressions in the fiber bundle  $\mathbb{T}^2 \rightarrow \bar{P}_t \rightarrow \bar{N}_t$  of the natural basis of  $H^1(\mathbb{T}^2, \mathbb{Z})$  corresponding to the splitting  $\mathbb{T}^2 = \text{diag}(e^{i\theta}, e^{i\psi}) \subset S^1 \times S^3$ . This is the basis of  $H^2(\bar{N}_t, \mathbb{Z})$  used in [KS1]. Notice though that in their notation  $a$  and  $b$  need to be switched (and hence  $n$  and  $m$  as well).

## 7. COMPARISON OF INVARIANTS

In this Section we discuss various diffeomorphisms that one obtains by comparing Kreck-Stolz invariants. Our main interest are diffeomorphisms with positively curved Eschenburg spaces, which can only exist for the manifolds  $S_{a,b}$ ,  $M_{a,b}^t$  and some of the  $\bar{M}_{a,b}^t$  ( $b$  even). Finally we also discuss diffeomorphism between various manifolds that admit Einstein metrics.

**7.1. Sphere bundles  $S_{a,b}$ .** In order to find sphere bundles  $S_{a,b}$  diffeomorphic to a positively curved Eschenburg space we use the following strategy. As the Kreck-Stolz invariants for the Eschenburg spaces are quite complicated we compare sphere bundles  $S_{a,b}$  to some fixed Eschenburg space  $E_{k,l}$ . We specify the invariants of  $E_{k,l}$  as follows:

$$r = |\sigma_2(k) - \sigma_2(l)| \ ; \ s_i(E_{k,l}) \equiv \frac{A_i}{B_i} \pmod{1} \ ; \ E_1 := \frac{224r}{B_1} A_1 \ ; \ E_2 := \frac{24r}{B_2} A_2 \ ; \ E_3 := \frac{6r}{B_3} A_3$$

where  $A_i, B_i \in \mathbb{Z}$ ,  $(A_i, B_i) = 1$  for  $i = 1, 2, 3$ .

We now describe under which conditions they are diffeomorphic to sphere bundles.

**THEOREM 7.1.** *A positively curved Eschenburg space  $E_{k,l}$  is diffeomorphic to an  $S^3$  bundle over  $\mathbb{C}\mathbb{P}^2$  if and only if:*

- (a)  $224r \equiv 0 \pmod{B_1}$  ;  $24r \equiv 0 \pmod{B_2}$  and  $6r \equiv 0 \pmod{B_3}$ .
- (b)  $E_1, E_2, E_3 + 1$  are even integers, and  $E_3 - E_2 - 3 \equiv 0 \pmod{3r}$ .
- (c)  $r + E_1$  has a square root mod  $224r$  such that  $[r + E_1]_{224r}^{1/2} + E_2 - 1 \equiv 0 \pmod{8r}$ .

If these conditions are satisfied,  $E_{k,l}$  is diffeomorphic to  $S_{a,a-r}$  if and only if

$$a \equiv \frac{r + 15[r + E_1]_{224r}^{1/2}}{2} + 7E_2 - 8 \pmod{168r}$$

for any square root satisfying (c). If  $r$  is negative, the diffeomorphism is orientation reversing.

*Proof.* Using Proposition 3.3 and Theorem 2.5, it follows that  $E_{k,l}$  is diffeomorphic to  $S_{a,a-r}$  if and only if

$$(7.2) \quad \begin{aligned} \frac{A_1}{B_1} - \frac{(2a-r+2)^2 - r}{224r} &=: x \in \mathbb{Z} \\ \frac{A_2}{B_2} + \frac{(2a-r+1)}{24r} &=: y \in \mathbb{Z} \\ \frac{A_3}{B_3} + \frac{(2a-r-2)}{6r} &=: z \in \mathbb{Z} \end{aligned}$$

This implies in particular the divisibility condition in part (a). We rewrite (7.2) as follows:

$$(7.3) \quad \begin{aligned} (2a-r+2)^2 &= E_1 + r - 224rx \\ 2a-r+1 &= -E_2 + 24ry \\ 2a-r-2 &= -E_3 + 6rz \end{aligned}$$

Since  $r$  is odd for an Eschenburg space and since  $a$  is an integer, this implies that  $E_1, E_2, E_3+1$  are even and that  $E_1+r$  has a square root mod  $224r$ . A solution to the second equation in (7.3) is a solution to the third equation if and only if  $6r(4y-z) = 3 + E_2 - E_3$ . Thus, if the divisibility condition in part (b) is satisfied, we can find an integer  $z$  for any integer solution  $y$ . We therefore set

$$a = -\frac{1}{2}(E_2 - r + 1) + 12ry.$$

Next we observe that  $e^2 \equiv f^2 \pmod{4n}$  implies that  $e \equiv \pm f \pmod{2n}$ . Thus, if we let  $S$  be a particular choice of a square root of  $E_1 + r \pmod{224r}$ , (7.3) implies that

$$1 - E_2 + 24ry = S + 112rx'$$

for some  $x' \in \mathbb{Z}$  and hence

$$S + E_2 - 1 = 8r(3y - 14x').$$

This implies the divisibility condition in part (c) and if we let  $\alpha = \frac{1}{8r}(S + E_2 - 1)$ , we get  $3y = \alpha + 14x'$  or  $y = 5\alpha + 14x''$ . Thus

$$a = -\frac{1}{2}(E_2 - r + 1) + 60r\alpha + 168rx'' = -\frac{1}{2}(E_2 - r + 1) + \frac{15}{2}(S + E_2 - 1) + 168rx''$$

which finishes our proof. □

Since there are infinitely many Eschenburg spaces (not necessarily positively curved) for a given  $r$ , it is conceivable that every  $\mathbb{S}^3$  bundle over  $\mathbb{C}\mathbb{P}^2$  is diffeomorphic to some Eschenburg space. But in [CEZ] it was shown that for a given  $r$ , there are only finitely many positively curved Eschenburg spaces. Our main interest are diffeomorphisms of sphere bundles with positively curved Eschenburg spaces. We use the computer program in [CEZ] to compute the invariants  $\frac{A_i}{B_i}$  for an Eschenburg space and write another program to find spaces satisfying the conditions in Theorem 7.1. A preliminary strong restriction is used in a search, since the computation of the invariants  $\frac{A_i}{B_i}$  is time consuming. According to

Corollary 3.4, a sphere bundle has standard linking form. This implies that only Eschenburg spaces with  $\sigma_3(k) - \sigma_3(l) = \pm 1 \pmod r$  can possibly be sphere bundles. This turns out to be a very strong restriction.

The invariants for Aloff-Wallach spaces have much simpler expressions and there are many solutions where  $W_{p,1}$  is diffeomorphic to a sphere bundle, in addition to the natural diffeomorphisms in Proposition 5.10. For example, for  $W_{1,1}$  one has  $s_1 = \frac{1}{112}$ ,  $s_2 = \frac{-1}{36}$ ,  $s_3 = \frac{1}{18}$  and hence  $E_1 = 6$ ,  $E_2 = -2$ ,  $E_3 = 1$ . There are 8 square roots of  $E_1 + r = 9 \pmod{224r}$ , but only the values 3 and 627 satisfy the divisibility condition. For  $r < 0$  there are no solutions. Thus  $W_{1,1}$  is diffeomorphic to a sphere bundle  $S_{a,a-r}$  if and only if  $a \equiv 2$  or  $146 \pmod{504}$ , whereas only the diffeomorphism with  $S_{2,-1}$  is a natural one.

There are also Aloff-Wallach spaces other than  $W_{p,1}$  which are diffeomorphic to a sphere bundle. For example,  $W_{56,103}$ , and thus  $r = 19513$ , is diffeomorphic to  $S_{a,a-r}$  if and only if  $a \equiv 273181 \pmod{3278184}$ .

Among the 14388 positively curved Eschenburg spaces for  $r < 1000$  there are, in addition to the 19 Aloff-Wallach spaces  $W_{p,1}$ , 13 which satisfy the conditions in Theorem 7.1, see Table A. We can now turn this around and state that among sphere bundles with  $r < 1000$ , only the ones listed in Table A are diffeomorphic to a positively curved Eschenburg space.

In the Tables an entry  $r^*$  indicates that the diffeomorphism is orientation reversing.

**7.2. Circle bundles  $M_{a,b}^t$ .** In this case one is not able to obtain a simple characterization of which Eschenburg spaces are diffeomorphic to such circle bundles since the invariants in Proposition 4.4 are too complicated. As we saw before such diffeomorphisms can only exist when  $t$  is odd since  $\pi_4(M_{a,b}^{2t}) = \mathbb{Z}_2$ . We thus limit the search to sample diffeomorphisms. Recall though that there are a number of natural diffeomorphisms of  $M_{a,b}^t$  with the positively curved Aloff-Wallach spaces. There are also many other diffeomorphisms with Aloff-Wallach spaces. For example,  $W_{1,1}$  is diffeomorphic to  $M_{a,b}^t$  with  $[a, b, t] = [-11, 2, 35]$ ,  $[-21, 1, 503]$  or  $[-25, 1, 647]$ . Similarly, for the circle bundles  $F_{p,q}$ . For example,  $F_{3,1}$  is diffeomorphic to  $M_{a,b}^t$  with  $[a, b, t] = [-189, 4, 2281]$  or  $[-111, 4, 799]$ .

To find other diffeomorphisms we fix bounds  $A, B$ , produce a list of all circle bundles  $M_{a,b}^t$  with  $r < A$  and  $|a|, |b| < B$  (letting  $t$  become large) and compute their Kreck-Stolz invariants. Similarly, for all positively curved Eschenburg spaces with  $r < A$ . We then compare the two lists to produce diffeomorphic pairs. In case where the bounds are  $r \leq 101$ ,  $|a|, |b| \leq 1000$ , there are a total of 316 diffeomorphisms which are not of natural type, 301 are with Aloff-Wallach spaces, 10 with the bundles  $F_{p,q}$  and the remaining 5 are listed in Table B.

Although  $\bar{M}_{a,b}^t$  is also a spin manifold, a Maple search did not produce any examples which are diffeomorphic to positively curved Eschenburg spaces.

**7.3. Einstein manifolds.** Among the manifolds we discussed, there are 3 classes which are known to admit Einstein metrics:

- [W] The homogeneous Aloff-Wallach spaces  $M_{a,b}^1 = W_{a,b}$ .
- [WZ] The circle bundles  $\bar{M}_{a,b}^0 = L_{a,b}$  with base  $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^1$ .
- [Che] The 3-sphere bundles  $S_{a,b}, \bar{S}_{a,b}$  whose structure group reduces to  $T^2 \subset \text{SO}(4)$ .



From the discussion in Section 1, it follows that a reduction to  $T^2$  is possible if and only if  $p_1(P_-)$  and  $p_1(P_+)$  are squares. To relate our notation to the one in [Che], recall that under the two fold cover a circle of slope  $(a, b)$  in  $S^3 \times S^3$  is sent to a circle of slope  $(a - b, a + b)$  in  $SO(4)$ . This easily implies that the manifolds in [Che], parametrized by  $q_1, q_2 \in \mathbb{Z}$  in his notation, are the manifolds  $S_{a,b}$  with  $4a + 1 = (q_1 + q_2)^2$  and  $4b + 1 = (q_1 - q_2)^2$  in the case of  $q_1 + q_2$  odd, and the manifolds  $\bar{S}_{a,b}$  with  $4a = (q_1 + q_2)^2$  and  $4b = (q_1 - q_2)^2$  in the case of  $q_1 + q_2$  even. In both cases  $r = a - b = q_1 q_2$ . For convenience, we denote these Einstein manifolds by  $C_{q_1, q_2}$  if  $q_1 + q_2$  is odd, and  $\bar{C}_{q_1, q_2}$  if  $q_1 + q_2$  is even.

Comparisons of Einstein manifolds within each class have been carried out before. In the second and third case the diffeomorphism classification is given by simple congruences, see [KS1, Che], and each space is diffeomorphic to infinitely many other Einstein manifolds. For example:

- The Einstein manifold  $L_{a,b}$  is diffeomorphic to  $L_{a,b'}$  if  $b \equiv b' \pmod{56a^2}$ .
- The Einstein manifold  $C_{q_1, q_2}$  (resp.  $\bar{C}_{q_1, q_2}$ ) is diffeomorphic to  $C_{q'_1, q'_2}$  (resp.  $\bar{C}_{q'_1, q'_2}$ ) if  $r = q_1 q_2 = q'_1 q'_2$  and  $q_1^2 + q_2^2 \equiv q'^2_1 + q'^2_2 \pmod{672r}$ .

For the Aloff-Wallach spaces on the other hand diffeomorphisms among each other are very rare, see [KS2].

Using our results, we can compare Einstein manifolds that belong to different classes. We first observe:

- There are no diffeomorphisms between the spin Einstein manifolds  $C_{q_1, q_2}$  and either  $W_{a,b}$  or  $L_{a,2b}$  since in the first case  $r$  is even, and in the other two cases  $r$  is odd.
- There are no diffeomorphisms between the spin Einstein manifolds  $W_{a,b}$  and  $L_{a,2b}$  since in the first case  $p_1 = 0$  and in the second case  $p_1 = 3(2b)^2 \pmod{a^2}$  which can never be 0 since  $(a, b) = 1$  (see Proposition 4.3 and Proposition 6.5).

But among the non-spin Einstein manifolds  $\bar{C}_{q_1, q_2}$  and  $L_{a,2b+1}$  there are some diffeomorphisms:

- The Einstein manifold  $\bar{C}_{q,q} \simeq \bar{S}_{q^2, 0}$  is naturally diffeomorphic to the Einstein manifold  $L_{q,1}$  (see Proposition 6.10).
- The Einstein manifold  $\bar{C}_{10, 490} \simeq \bar{S}_{62500, 57600}$ , is diffeomorphic to the Einstein manifold  $L_{70, 5899}$ .

Further examples are difficult to find. Indeed, if a diffeomorphism between  $C_{q_1, q_2}$  and  $L_{a,b}$  exists, then  $q_1 q_2$  must be a square, and the linking form of  $L_{a,b}$  must be standard. Both of these are strong restrictions.

We finally remark that the spin Einstein manifolds  $C_{q_1, q_2}$  can never be diffeomorphic to an Eschenburg space since in the first case  $q_1 + q_2$  is odd, and hence  $r = q_1 q_2$  is even, whereas for Eschenburg space  $r$  is always odd. Notice also that  $L_{a,b}$  can never be diffeomorphic to an Eschenburg space since  $\pi_4(L_{a,b}) = \mathbb{Z}_2$ .

For the convenience of the reader, the Maple program that computes the invariants is available at [www.math.upenn.edu/wziller/research](http://www.math.upenn.edu/wziller/research).

| $r$  | $[k_1, k_2, k_3 \mid l_1, l_2, l_3]$ | $[a] \bmod 168r$   | $s_1$           | $s_2$        | $s_3$       |
|------|--------------------------------------|--------------------|-----------------|--------------|-------------|
| 41*  | $[2, 3, 7 \mid 12, 0, 0]$            | $[2285, 5237]$     | $115/287$       | $65/164$     | $-33/82$    |
| 127  | $[17, 16, -7 \mid 14, 12, 0]$        | $[17230]$          | $3489/14224$    | $-403/1524$  | $-41/762$   |
| 233  | $[5, 3, -31 \mid -23, 0, 0]$         | $[2943, 36495]$    | $-1863/6524$    | $-31/2796$   | $-59/1398$  |
| 289* | $[21, 18, -13 \mid 16, 10, 0]$       | $[21194, 42002]$   | $-397/1156$     | $121/1734$   | $481/1734$  |
| 611  | $[25, 17, -23 \mid 14, 5, 0]$        | $[69423, 84087]$   | $-15789/68432$  | $-1565/3666$ | $1075/3666$ |
| 617  | $[24, 19, -23 \mid 14, 6, 0]$        | $[13030]$          | $-3567/8638$    | $1043/3702$  | $473/3702$  |
| 661  | $[23, 21, -26 \mid 18, 0, 0]$        | $[56346, 72210]$   | $1787/37016$    | $-41/661$    | $-327/1322$ |
| 673* | $[25, 14, -25 \mid 8, 6, 0]$         | $[49154, 81458]$   | $-529/2692$     | $181/4038$   | $721/4038$  |
| 751* | $[33, 33, -20 \mid 26, 20, 0]$       | $[7036, 43084]$    | $-12629/84112$  | $-2351/9012$ | $-199/4506$ |
| 911* | $[69, 65, -13 \mid 63, 58, 0]$       | $[123457, 145321]$ | $-7445/102032$  | $1375/5466$  | $31/5466$   |
| 911* | $[23, 23, -31 \mid 14, 1, 0]$        | $[1457, 132641]$   | $31083/102032$  | $167/1822$   | $667/1822$  |
| 929* | $[41, 17, 4 \mid 62, 0, 0]$          | $[22359, 44655]$   | $1441/6503$     | $401/11148$  | $805/5574$  |
| 991* | $[51, 45, -19 \mid 43, 34, 0]$       | $[18113, 89465]$   | $-44333/110992$ | $2863/5946$  | $-443/5946$ |

TABLE A. Sphere bundles  $S_{a,a-r}$  with  $r < 1000$  which are diffeomorphic to positively curved Eschenburg Spaces  $E_{k,l}$  other than  $W_{p,1}$ .

| $r$ | $[k_1, k_2, k_3 \mid l_1, l_2, l_3]$ | $[a, b, t]$            | $s_1$      | $s_2$      | $s_3$    |
|-----|--------------------------------------|------------------------|------------|------------|----------|
| 17  | $[1, 2, 5 \mid 8, 0, 0]$             | $[638, -607, -403]$    | $-201/952$ | $55/204$   | $23/102$ |
| 25  | $[1, 2, -9 \mid -6, 0, 0]$           | $[621, -614, -7781]$   | $19/50$    | $-3/10$    | $-17/50$ |
| 33  | $[1, 1, 16 \mid 18, 0, 0]$           | $[805, -632, -17]$     | $47/308$   | $-125/396$ | $53/198$ |
| 41* | $[2, 3, 7 \mid 12, 0, 0]$            | $[580, -579, -335861]$ | $-115/287$ | $-65/164$  | $33/82$  |
| 41* | $[2, 3, 7 \mid 12, 0, 0]$            | $[405, -404, -163661]$ | $-115/287$ | $-65/164$  | $33/82$  |

TABLE B. Circle bundles  $M_{a,b}^t$ , other than  $W_{p,q}$ ,  $F_{p,q}$ , diffeomorphic to positively curved Eschenburg Spaces  $E_{k,l}$ .

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