1. Problem 7.31: Fix $z$ in $\mathbb{C}$ and $r > |z|$. Prove that $\sum_{k \geq 0} \left(\frac{z}{w}\right)^k$ converges uniformly in $w$, for $|w| \geq r$.

Note that

$$\left|\frac{z}{w}\right| < \frac{r}{|w|} \leq 1, \text{ if } |w| \geq r.$$ 

In particular, for all $w$ with $|w| \geq r$,

$$\left|\frac{z}{w}\right| \leq \left|\frac{z}{r}\right| < 1.$$ 

Since $\sum_{k \geq 0} \frac{z}{r}$ is a geometric series with limit $\frac{1}{1-r}$, it follows from the Weierstraß-$M$-test that $\sum_{k \geq 0} \left(\frac{z}{w}\right)^k$ converges uniformly in $w$, for $|w| \geq r$.

2. Problem 7.37: Define $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}$ via $f_n(t) = \frac{1}{n} e^{-\frac{t}{n}}$ for $n \geq 1$.

- Show that the maximum of $f_n(t)$ is $\frac{1}{n}$. Since for all $n \geq 1$, $d/dt(f_n(t)) = -(1/n^2) e^{-t/n} < 0$ for all $t \geq 0$, the functions $f_n(t)$ are decreasing in $t$, hence attain their maximum at $t = 0$. This maximum is $1/n$.

- Show that $f_n(t)$ converges uniformly to the zero function on $\mathbb{R}_{\geq 0}$. Fix $\epsilon > 0$, and choose an integer $N$ large enough such that $N > 1/\epsilon$. Then for all $n \geq N$, the previous item implies that:

$$|f_n(t) - 0| = |f_n(t)| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon, \text{ for all } t \geq 0,$$

and thus $f_n(t)$ converges uniformly to the zero function as $n \to \infty$.

- Show that $\int_0^\infty f_n(t) dt$ does not converge to 0 as $n \to \infty$. We have that for all $n \geq 1$:

$$\int_0^\infty f_n(t) dt = - e^{-\frac{1}{n}} \big|_{t=0}^{t=\infty} = 1$$

- This does not contradict Proposition 7.27 because the domain of integration $[0, \infty)$ is not a piecewise smooth path. Indeed, a piecewise smooth path is necessarily bounded.

*Patrick De Leenheer, email: deleenhp@math.oregonstate.edu*
3. Prove that if \( f \) is entire and \( \text{Im}(f) \) is constant on the closed unit disk, then \( f \) is constant. First, let \( f = u + iv \), and apply the Cauchy-Riemann equations to points \((x_0, y_0)\) in the open unit disk:

\[
\begin{align*}
    u_x(x_0, y_0) & = v_y(x_0, y_0) \equiv 0 \\
    u_y(x_0, y_0) & = -v_x(x_0, y_0) \equiv 0,
\end{align*}
\]

where we’ve used that \( v \) is constant on the unit disk. Hence \( u \) is also constant on the open unit disk, and therefore \( f(z) = C \) for all \( z \) with \( |z| \leq 1 \), where \( C \) is some constant. Now define \( g(z) = f(z) - C \). Then \( g \) is also entire, and \( g(z) = 0 \) on the unit disk. By the identity principle follows that \( g(z) = 0 \) for all \( z \) in \( \mathbb{C} \). But then \( f(z) = C \) for all \( z \) in \( \mathbb{C} \).

4. Suppose that \( f \) is holomorphic at \( z_0 \), \( f(z_0) = 0 \), and \( f'(z_0) \neq 0 \). Show that \( f \) has a zero of multiplicity 1 at \( z_0 \). Clearly, \( f \) has a zero at \( z_0 \), and since \( f'(z_0) \neq 0 \), and using the classification of zeros, \( f \) cannot be identically zero on some open disk centered at \( z_0 \) (otherwise \( f'(z_0) \) would equal zero). Thus \( f \) has an isolated zero of some finite multiplicity \( m \geq 1 \) at \( z_0 \), and there exists a holomorphic function \( g : D(z_0, R) \to \mathbb{C} \) with \( R > 0 \) such that \( g(z_0) \neq 0 \), and such that

\[
f(z) = (z - z_0)^m g(z), \quad \text{if } z \in D(z_0, R).
\]

But then

\[
0 \neq f'(z_0) = m(z - z_0)^{m-1}g(z)|_{z=z_0},
\]

which implies that \( m = 1 \), for if \( m > 1 \), the right-hand side would be zero.