8.2#38

$$\frac{2x}{1+x^2} = 2x\cdot\frac{1}{1-(-x^2)} = 2x\cdot\left(1+(-x^2)+(-x^2)^2+\dots\right), \quad |x^2|<1 \text{ or } |x|<1$$

$$= 2x\sum_{n=0}^{\infty}(-x^2)^n = 2x\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}2(-1)^n x^{2n+1}, \quad |x|<1$$

$$\int_0^x \frac{2u}{1+u^2}\,du = \ln(1+x^2)+0 \Rightarrow \ln(1+x^2) = \int_0^x \sum_{n=0}^{\infty}2(-1)^n x^{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty}\cancel{2}(-1)^n\frac{x^{2(n+1)}}{\cancel{2}(n+1)}$$

$$\ln(1+x^2) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2(n+1)}}{(n+1)}, \quad |x|<1$$

8.2#34 $\quad y''+\frac{2}{x}y'+y^n=0,\ y(0)=1,\ y'(0)=0$. Assume $n>1$.

$$(*)\ y = 1+\underbrace{}_{\text{missing } x \text{ term}}c_2x^2+c_3x^3+\dots \Rightarrow y^n = \left(1+c_2x^2+c_3x^3+\dots\right)^n = \left(\sum_{i=0}^{\infty}c_ix^i\right)^n \text{ with } c_0=1,\ c_1=0$$

since $y(0)=1$, $y'(0)=0$

For any power series, the coefficients can be calculated in terms of derivatives evaluated at the point at which the series is centered.

Let $y^n = \sum_{i=0}^{\infty}a_ix^i$. Then $a_i = \frac{1}{i!}\frac{d^i}{dx^i}(y^n)\Big|_{x=0}$, $i=0,1,\dots$

$$\Rightarrow \begin{cases} a_0 = \frac{1}{1}\cdot 1 \\ a_1 = \frac{1}{1}\cdot\frac{d}{dx}(y^n)\Big|_{x=0} = ny^{n-1}\frac{d}{dx}(y)\Big|_{x=0} \overset{(*)}{=} n\cdot 1\cdot 0 = 0 \\ a_2 = \frac{1}{2}\cdot\frac{d^2}{dx^2}(y^n)\Big|_{x=0} = \frac{1}{2}\frac{d}{dx}\left(ny^{n-1}\frac{dy}{dx}\right)\Big|_{x=0} = \frac{n}{2}\left((n-1)y^{n-2}\left(\frac{dy}{dx}\right)^2+y^{n-1}\frac{d^2y}{dx^2}\right) \\ \overset{(*)}{=} \frac{n}{2}\left(0+1\cdot 2c_2\right) = nc_2 \end{cases}$$

Thus, $y^n = 1+nc_2x^2+\dots$ Plug into equation:

$$x\left(2c_2+6c_3x+12c_4x^2+\dots\right)+2\left(2c_2x+3c_3x^2+4c_4x^3+\dots\right)+x\left(1+nc_2x^2+\dots\right)=0$$

coeff. of x: $2c_2+4c_2+1=0 \rightarrow c_2=-1/6$

x^2: $6c_3+6c_3+0=0 \rightarrow c_3=0$

x^3: $12c_4+8c_4+nc_2=0 \rightarrow c_4 = -\frac{nc_2}{20} = +\frac{n}{20\cdot 6} = \frac{n}{120}$

$$\Rightarrow y = 1-\frac{1}{6}x^2+\frac{n}{120}x^4+\dots$$