Notes on matrices and systems

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1 Matrices

Matrices are tables of generally complex numbers. Examples:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Matrix A is said to be *square* because it has equally many rows and columns, namely 2. Matrix B on the other hand is *rectangular* and has 2 rows and 3 columns.

A general matrix X may have n rows and m columns, and we sometimes write $X \in \mathbb{R}^{n\times m}$ if all entries of the matrix are real numbers, or $X \in \mathbb{C}^{n\times m}$ if they are complex. Special matrices are *row vectors* ($n = 1$) and *column vectors* ($m = 1$).

Operations on matrices

We can **add** matrices X and Y if (and only if) both have the same number of rows and columns, and this is done entrywise:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix}$$

We **cannot** add the matrices A and B defined earlier.

We can **multiply** matrix X and Y (if and only if) matrix X has the same number of columns as the matrix Y has rows, i.e. if and only if $X \in \mathbb{C}^{n\times m}$ and $Y \in \mathbb{C}^{m\times p}$. The result of the multiplication is a matrix $Z = XY \in \mathbb{C}^{n\times p}$, i.e. it has the same number of rows as X and the same number of columns as Y. So what is Z? Let's specify each of its entries. Denoting $[Z]_{ij}$ as the entry in the ith row and jth column of Z, we have that

$$[Z]_{ij} = \sum_{k=1}^{m} [X]_{ik}[Y]_{kj}, \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, p.$$

Notice that this formula is also given by the *dot product of the ith row of X and the jth column of Y*. For example, we can calculate AB (but **not** BA; why not?) and AA which we denote for short as A^2:

$$AB = \begin{pmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \end{pmatrix}, A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

We can **multiply a matrix by a scalar**: If $X \in \mathbb{C}^{n\times m}$ and $\alpha \in \mathbb{C}$ then $[\alpha X]_{ij} = \alpha [X]_{ij}$. For example,

$$2A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}.$$

The following rules are easily verified, provided that the operation makes sense. The matrix 0 is a matrix having only zero entries, and the matrix I_n is a matrix in $\mathbb{R}^{n\times n}$ with all diagonal entries equal to 1, and off-diagonal entries equal to 0: $[I_n]_{ii} = 1$ for all $i = 1, \dots, n$ and $[I_n]_{ij} = 0$

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if $i \neq j$.

$$\begin{aligned}(A+B)+C &= A+(B+C)\\ A+0 &= A=0+A\\ A+B &= B+A\\ A+(-A) &= 0=(-A)+A\\ (AB)C &= A(BC)\\ I_nA &= A=AI_m\\ A(B+C) &= AB+AC\end{aligned}$$

AB and BA are not necessarily the same, even if both exist (can you give an example?).

Determinants: Given a square matrix $X \in \mathbb{C}^{n\times n}$, we can associate a complex number to it, denoted as $\det A$. **I am not giving the general definition which requires some linear algebra**, but here is how you compute the determinant for matrices in $\mathbb{C}^{2\times 2}$ and $\mathbb{C}^{3\times 3}$:

$$\det\begin{pmatrix} a & b\\ c & d\end{pmatrix} = ad-bc,$$

$$\det\begin{pmatrix} x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{pmatrix} = x_{11}x_{22}x_{33}+x_{12}x_{23}x_{31}+x_{13}x_{32}x_{21}-x_{13}x_{31}x_{22}-x_{21}x_{12}x_{33}-x_{11}x_{23}x_{32}$$

We say that X is **singular** if and only if $\det X = 0$, and **non-singular** otherwise.

If for given $X \in \mathbb{C}^{n\times n}$, there is some $Y \in \mathbb{C}^{n\times n}$ such that $XY = I_n = YX$, then we say that X is **invertible with inverse** Y. If X is invertible, then its inverse is unique (can you prove this?), and we usually denote it as X^{-1}. Matrices are not always invertible. In fact, it can be shown that a matrix is invertible if and only if it is non-singular. Here's how you calculate the inverse of an invrtible 2-by-2 matrix:

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$$

(To convince yourself that this is the correct result, multiply the matrix and its inverse; is the result equal to I_2?)

Eigenvalue-eigenvector pairs Given $A \in \mathbb{C}^{n\times n}$, we say that $\lambda \in \mathbb{C}$ is *an eigenvalue of* A if there is some vector $x \in \mathbb{C}^{n\times 1}$ with $x \neq 0$ such that:

$$Ax = \lambda x. \tag{1}$$

The vector x is called *an eigenvector of* A.

How to find eigenvalues and eigenvectors? This proceeds in two steps. First, we find the eigenvalues. Rewrite (1):

$$(A-\lambda I_n)x = 0. \tag{2}$$

Demanding that the above system of linear equations in the unknown x, has a **non-zero solution**, is equivalent to asking that the matrix $(A-\lambda I_n)$ is singular (this would be proved in a typical linear algebra course). This in turn is equivalent to requiring that:

$$\det(A-\lambda I_n) = 0. \tag{3}$$

This is a polynomial equation in λ of degree n, called the **characteristic equation**. Example:

$$A = \begin{pmatrix} 1 & 2\\ 2 & 1\end{pmatrix}$$

Then the characteristic equation

$$\det(A-\lambda I_2) = \det\begin{pmatrix} 1-\lambda & 2\\ 2 & 1-\lambda\end{pmatrix} = 1-2\lambda+\lambda^2-4 = \lambda^2-2\lambda-3 = 0,$$

is a quadratic equation with solutions $\lambda_1 = 3$ and $\lambda_2 = -1$.

The second step is to find, for every eigenvalue, a corresponding eigenvector, i.e., a non-zero solution x for (2). Continuing the example, let's find an eigenvector x_1 corresponding to $\lambda_1 = 3$. This is a solution to

$$\begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix} x_1 = 0,$$

and

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is clearly a solution (so is any non-zero scalar multiple of x_1; check this. Actually, this is a general fact, true for any eigenvector). Similarly, for $\lambda_2 = -1$, the vector

$$x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

is a corresponding eigenvector. (verify this)

Remark 1. MATLAB is available on all campus computers. It is software that is very friendly to matrices. When you start it up, you will see a command line >> appear. To define a matrix

$$\begin{pmatrix} 1 & 1.2 \\ 3 & 4 \end{pmatrix},$$

type

$$A = [1 \;\; 1.2 \;;\; 3 \;\; 4]$$

after the command line and hit return. MATLAB will produce an output in the form of the desired matrix. Now that this matrix has been defined, you can calculate its eigenvector-eigenvalues pairs by typing the following:

[T,D]=eig(A)

MATLAB will return two matrices T and D, where T contains the eigenvectors and D is a diagonal matrix containing the eigenvalues on the diagonal.

2 Discrete time linear systems

We define a **discrete time linear system** on $\mathbb{C}^n$ as follows:

$$x(t+1) = Ax(t), \quad t = 0, 1, 2, \dots \tag{4}$$

where $x(t)$ is a column vector denoting the state of some physical system. It is called the state vector and it belongs to $\mathbb{C}^n$. [1]

Given the value of $x(0)$ (the initial condition), we can calculate the future values of x by matrix multiplication:

$$x(1) = Ax(0), x(2) = Ax(1) = A^2x(0), \dots$$

It is easy to see, that for an aribitrary $t = 0, 1, 2, \dots$ (if we agree that $A^0 = I_n$)

$$x(t) = A^t x(0), \tag{5}$$

showing that it is important to be able to calculate the powers of a square matrix. Is there a convenient way to do this?

Matrix powers In the special case that $A \in \mathbb{C}^{n\times n}$ is a diagonal matrix:

$$A = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix},$$

[1] In this course most systems will be defined on $\mathbb{R}^n$, rather than on $\mathbb{C}^n$. Then $x(t)$ belongs to $\mathbb{R}^n$ and A to $\mathbb{R}^{n\times n}$.

calculating A^t for any $t = 0, 1, 2, \dots$ is easy because the powers A^t are also diagonal:

$$A^t = \begin{pmatrix} d_1^t & 0 & \dots & 0 \\ 0 & d_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n^t \end{pmatrix}$$

Is this of any use when we wish to calculate the powers of a matrix that isn't diagonal? The answer is yes!

First, we say that a matrix $A \in \mathbb{C}^{n\times n}$ is **diagonalizable** if there is some non-singular matrix $T \in \mathbb{C}^{n\times n}$ such that

$$T^{-1}AT = D,$$

where D is some diagonal matrix.

Let's see why this helps us to calculate A^t?

$$A^t = (TDT^{-1})^t = (TDT^{-1})(TDT^{-1})\dots(TDT^{-1}) = TD^tT^{-1}, \tag{6}$$

and since D^t is diagonal as we pointed out earlier, the final matrix, TD^tT is very easily calculated as the product of three matrices!

Of course, the question is how to determine if a given matrix A is diagonalizable, and if so, how to find the matrices T and D. This is a very important question studied at depth in a linear algebra course. The answer is as follows:

Theorem 1. *A matrix $A \in \mathbb{C}^{n\times n}$ is diagonalizable if and only if it has a basis of eigenvectors. If A is diagonalizable, then D is the diagonal matrix that contains the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ on the diagonal, and T is the matrix formed by writing the corresponding eigenvectors $x_1, x_2, \dots, x_n$.*

But what is a basis? A *basis* (for $\mathbb{C}^n$) is a set of n vectors $\{z_1, \dots, z_n\}$ with the property that any vector $x \in \mathbb{C}^n$ can be written as a linear combination of the vectors $z_1, \dots, z_n$. That is, there must exist complex scalars $c_1, \dots, c_n$, such that

$$x = c_1z_1 + c_2z_2 + \dots + c_nz_n = Zc, \text{ where } Z = [z_1\ z_2\ \dots\ z_n] \text{ and } c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

To test if a given set of vectors $\{z_1, \dots, z_n\}$ is a basis, it suffices to check that the matrix Z is non-singular, or equivalently, that its determinant is non-zero.

It turns out that many matrices are diagonalizable. For instance, every matrix that has **distinct** eigenvalues is diagonalizable (this is shown in a linear algebra course).

Let us illustrate this on our previous example. The eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$, hence A is diagonalizable, i.e. $T^{-1}AT = D$ with

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(Note that T is indeed non-singular: $\det T = 2$.) Consequently, the powers of A are

$$A^t = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & (-1)^t \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3^t + (-1)^t & 3 - (-1)^t \\ 3 - (-1)^t & 3^t + (-1)^t \end{pmatrix}$$

I hope you see that evaluating A^{1000} is a lot easier using this formula, than multiplying A 999 times with itself.

3 Stability for discrete time linear systems

Let us start with a very simple case that reveals the key issues concerning the notion of stability. Consider a discrete time linear system, defined on $\mathbb{C}$;

$$x(t+1) = ax(t), \quad t = 1, 2, \dots$$

for some $a \in \mathbb{C}$.

Given the initial condition $x(0)$ we wonder what happens to the solution sequence $x(1), x(2), \dots$ when $t \to \infty$. Does the sequence converge, or not? Does it remain bounded, or not? We know that

$$x(t) = a^t x(0), \quad t = 0, 1, \dots,$$

and therefore, using the notation $|a|$ for the modulus [2] of a:

1. If $|a| < 1$, then $\lim_{t\to+\infty} x(t) = 0$, no matter what $x(0)$ is.
2. If $|a|$=1, then $x(t)$ remains bounded for all t, no matter what $x(0)$ is: $|x(t)| = |a^t x(0)| = |x(0)|$.
3. If $|a| > 1$, then $\lim_{t\to\infty} x(t) = \infty$ when $x(0) \neq 0$. In particular, if a solution does not start in 0, it grows unbounded.

So it turns out that the deciding factor in this discussion is the **modulus of a**.

We would like to extend this to general discrete time linear systems (4). But what is the generalization of the modulus of a complex number to a matrix? It turns out that this is not the right question. Instead, we will see below that the results are stated in terms of **the moduli of the eigenvalues of the matrix A**.

Consider (4) and assume that the matrix A is diagonalizable. By Theorem 1 we know that A has eigenvalues $\lambda_1, \lambda_1, \dots, \lambda_n$ and a basis of associated eigenvectors $x_1, x_2, \dots, x_n$. Now consider the initial condition $x(0)$ for our system. Since the eigenvectors of A form a basis, we can write $x(0)$ as a linear combination of the eigenvectors:

$$x(0) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \tag{7}$$

for appropriately chosen complex scalars $c_1, c_2, \dots, c_n$. Then

$$x(t) = A^t x(0),$$

and combining this with (7), we find that

$$x(t) = A^t(c_1x_1+c_2x_2+\dots+c_nx_n) = c_1A^tx_1+c_2A^tx_2+\dots+c_nA^tx_n = c_1\lambda_1^t x_1+c_2\lambda_2^t x_2+\dots c_n\lambda_n^t x_n.$$

In the last step we repeatedly used that $Ax_i = \lambda_i x_i$ for all $i = 1, 2, \dots, n$. From this expression we immediately see that

Theorem 2. *Let $A \in \mathbb{R}^{n\times n}$ be diagonalizable, with eigenvalues λ_i for $i = 1, \dots, n$.*

1. *If $|\lambda_i| < 1$ for all $i = 1, \dots, n$, then $\lim_{t\to+\infty} x(t) = 0$, no matter what $x(0)$ is.*
2. *If $|\lambda_i| \leq 1$ for all $i = 1, \dots, n$, then $x(t)$ remains bounded.*
3. *If there is some eigenvalue λ_j such that $|\lambda_j| > 1$, then $x(t)$ grows unbounded for almost all initial conditions $x(0)$. More precisely, $x(t)$ grows unbounded whenever the initial condition $x(0)$ is such that $c_j \neq 0$ in (7).*

Visualizing the condition on the eigenvalues in the complex plane $\mathbb{C}$, it amounts to checking whether the eigenvalues of A are inside the unit circle $S = \{z \in \mathbb{C} \mid |z| = 1\}$ (case 1), on the unit disk $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ (case 2), or that A has an eigenvalue outside the unit disk (case 3).

We say that system (4) **is asymptotically stable in case** 1**, stable in case** 2 **and unstable in case** 3.

[2] Recall that the modulus of a complex number $z = \alpha + i\beta$ is defined as $|z| = \sqrt{\alpha^2 + \beta^2}$.

4 Continuous time linear systems

In this section we discuss continuous time linear systems:

$$\dot{x}(t) = Ax(t), \tag{8}$$

where $x(t) \in \mathbb{R}^n$, and $t \in \mathbb{R}$, and $\dot{x}(t)$ stands for $d/dt(x(t))$.

Do not confuse discrete and continuous time linear systems (4) **and** (8)**!**

Given an initial condition $x(0)$ for (8), can we find its solution $x(t)$, provided this solution exists? If $x(t) \in \mathbb{R}$ and $A = a \in \mathbb{R}$, then we know from our course in differential equations that $x(t) = \mathrm{e}^{at}\, x(0)$ (the differential equation is separable, and can be solved easily). But what if $A \in \mathbb{R}^{n\times n}$ and $n > 1$? This is also discussed in a typical differential equations course, and involves the concept of a *fundamental matrix solution*. We will not repeat that discussion here, but rather accept the following fact:

The solution $x(t)$ to (8) with initial condition $x(0)$ exists and is unique for all $t \in \mathbb{R}$, and it is given by:

$$x(t) = \mathrm{e}^{tA}\, x(0).$$

But what is e^{tA}? It is called the **matrix exponential**, and it is defined as follows (where we use the convention that $0! = 1$ and $A^0 = I_n$):

$$\mathrm{e}^{tA} = \sum_{i=0}^{\infty} \frac{1}{i!}(tA)^i = I_n + tA + \frac{1}{2!}(tA)^2 + \frac{1}{3!}(tA)^3 + \dots$$

It can be shown that this matrix series is well-defined for all $t \in \mathbb{R}$ (i.e., it converges; in fact, it converges absolutely).

Given A, how to calculate e^{tA}? For a general matrix, this might be quite hard using the definition directly, as you may suspect. An exception is the case where the matrix is diagonal. If

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix},$$

then

$$\mathrm{e}^{tD} = I_n + (tD) + \frac{1}{2!}(tD)^2 + \dots = \begin{pmatrix} \mathrm{e}^{td_1} & 0 & \dots & 0 \\ 0 & \mathrm{e}^{td_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathrm{e}^{td_n} \end{pmatrix}.$$

In other words, the matrix exponential of a diagonal matrix is also diagonal!

How should we proceed for a non-diagonal matrix? The answer lies in realizing that this is a problem of calculating the powers of a matrix, which, as we already know, is easy for diagonalizable matrices. So, assume that A is diagonalizable. Then there is a non-singular matrix T and a diagonal matrix D such that $A = TDT^{-1}$. Plug this into the definition of the matrix exponential:

$$\begin{aligned}
\mathrm{e}^{tA} &= I_n + T(tD)T^{-1} + \frac{1}{2!}(T(tD)T^{-1})^2 + \frac{1}{3!}(T(tD)T^{-1})^3 + \dots \\
&= I_n + T(tD)T^{-1} + \frac{1}{2!}(T(tD)T^{-1})(T(tD)T^{-1}) + \frac{1}{3!}(T(tD)T^{-1})(T(tD)T^{-1})(T(tD)T^{-1}) + \dots \\
&= TT^{-1} + \frac{1}{2!}T(tD)^2T^{-1} + \frac{1}{3!}T(tD)^3T^{-1} + \dots \\
&= T\left(I_n + (tD) + \frac{1}{2!}(tD)^2 + \frac{1}{3!}(tD)^3 + \dots\right)T^{-1} \\
&= T\,\mathrm{e}^{tD}\,T^{-1},
\end{aligned}$$

the product of three matrices which can be calculated easily, once you know tbe eigenvalue-eigenvector pairs of matrix A.

Example Continuing our example

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

we find that

$$\mathrm{e}^{tA} = T\,\mathrm{e}^{tD}\,T^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{3t} & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathrm{e}^{3t} + \mathrm{e}^{-t} & \mathrm{e}^{3t} - \mathrm{e}^{-t} \\ \mathrm{e}^{3t} - \mathrm{e}^{-t} & \mathrm{e}^{3t} + \mathrm{e}^{-t} \end{pmatrix} \tag{9}$$

Alternative method Another practical method to calculate matrix exponentials -often used in engineering- is based on the inverse Laplace transform:

$$\mathrm{e}^{tA} = \mathcal{L}^{-1}\left((sI_n - A)^{-1}\right).$$

Let us illustrate this on our example:

$$(sI_2 - A) = \begin{pmatrix} s-1 & -2 \\ -2 & s-1 \end{pmatrix} \Rightarrow (sI_2 - A)^{-1} = \begin{pmatrix} \frac{s-1}{(s+1)(s-3)} & \frac{2}{(s+1)(s-3)} \\ \frac{2}{(s+1)(s-3)} & \frac{s-1}{(s+1)(s-3)} \end{pmatrix}$$

Using partial fractions, we find that

$$(sI_2 - A)^{-1} = \begin{pmatrix} \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-3} & \frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-3} \\ \frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-3} & \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-3} \end{pmatrix},$$

and taking the inverse Laplace transform we find that

$$\mathrm{e}^{tA} = \begin{pmatrix} \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{3t} & -\frac{1}{2}\mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{3t} \\ -\frac{1}{2}\mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{3t} & \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{3t} \end{pmatrix},$$

which agrees with (9).

Some properties of matrix exponentials Let A and B be complex n by n matrices, and t, t_1, t_2 be real numbers. Then (no proofs):

$$\begin{aligned} \mathrm{e}^{t_1 A}\,\mathrm{e}^{t_2 A} &= \mathrm{e}^{(t_1+t_2)A} \\ \mathrm{e}^{tA} &\quad \text{is non-singular and } \left(\mathrm{e}^{tA}\right)^{-1} = \mathrm{e}^{-tA} \\ \mathrm{e}^{tA}\,\mathrm{e}^{tB} &= \mathrm{e}^{t(A+B)} \text{ if } AB = BA \text{ (i.e. if } A \text{ and } B \text{ commute)} \\ \frac{d}{dt}\mathrm{e}^{tA} &= A\,\mathrm{e}^{tA} = \mathrm{e}^{tA}\,A \end{aligned}$$

Note that $AB = BA$ is sufficient for the third property to hold, but not necessary (can you show this by constructing an example?).

Stability for continuous time linear systems Reconsider system (8), where A is assumed to be diagonalizable. If $x(0)$ is the initial condition, then we have seen that the solution $x(t)$ of (8) is described by the following formula:

$$x(t) = \mathrm{e}^{tA}\,x(0) = T\,\mathrm{e}^{tD}\,T^{-1}x(0),$$

we see that all components of $x(t)$ are linear combinations of exponential functions e^{td_i}. If $d_i < 0$, then $\mathrm{e}^{td_i} \to 0$ as $t \to +\infty$. If $d_i = 0$, then $\mathrm{e}^{td_i} = 1$ is bounded, while if $d_i > 0$, then $\mathrm{e}^{td_i} \to \infty$ as $t \to +\infty$. If d_i is a complex number[3], say $d_i = \alpha_i + i\beta_i$, then the same conclusions remain true if we replace d_i in the above inequalities by α_i, i.e. by the real part of d_i, denoted by $\mathcal{R}(d_i)$.

Theorem 3. *Let $A \in \mathbb{R}^{n\times n}$ be diagonalizable, with eigenvalues λ_i for $i = 1, \ldots, n$.*

1. *If $\mathcal{R}(\lambda_i) < 0$ for all $i = 1, \ldots, n$, then $\lim_{t\to\infty} x(t) = 0$, no matter what $x(0)$ is.*
2. *If $\mathcal{R}(\lambda_i) \leq 0$ for all $i = 1, \ldots, n$, then $x(t)$ remains bounded.*

[3]In that case $\mathrm{e}^{td_i} = \mathrm{e}^{t(\alpha_i + i\beta_i)} = \mathrm{e}^{t\alpha_i}(\cos(\beta_i t) + i\sin(\beta_i t))$ by Euler's formula.

3. *If there is some eigenvalue λ_j such that $\mathcal{R}(\lambda_j) > 0$, then $x(t)$ grows unbounded for almost all initial conditions $x(0)$.*

Notice in particular the difference between Theorem 2 and 3! Please do not confuse these results. The first applies to discrete time systems, and the second to continuous time systems.

Visualizing the condition on the eigenvalues in the complex plane $\mathbb{C}$, it amounts to checking whether the eigenvalues of A are in the open left-half plane $\{z \in \mathbb{C} | \mathcal{R}(z) < 0\}$ (case 1), in the closed left-half plane $\{z \in \mathbb{C} | \mathcal{R}(z) \leq 0\}$(case 2), or that A has an eigenvalue in the open right-half plane $\{z \in \mathbb{C} | \mathcal{R}(z) > 0\}$ (case 3).

As before, we say that system (8) **is asymptotically stable in case 1, stable in case 2 and unstable in case 3.**

5 Exercise

Let

$$A = \begin{pmatrix} -1 & 1 \\ 0 & a \end{pmatrix}$$

where a is a real parameter and $a \neq -1$.

1. Find A^{107}.
2. Find e^{tA}.
3. If $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for (4), find $x(t)$.
4. If $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for (8), find $x(t)$.
5. Discuss stability of system (4).
6. Discuss stability of system (8).
7. Why was $a \neq -1$ assumed?