Notes on a juvenile-adult model

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These notes describe a model of a population consisting of juveniles and adults. We will determine the fixed points and their stability.

We denote J_n and A_n as the number of juveniles and adults at time n respectively and make the following assumptions:

1. A fraction $f \in (0,1)$ of juveniles makes it to adulthood between two consecutive times. The rest, a fraction $1-f$, remain juveniles.

2. The average number of off-spring per adult between consecutive times is $b \in (0,1)$.

3. The survival chances of an adult between consecutive times depends on the current number of adults in the population (density dependence). For instance, food sources for adults could be scarce. If the population has A_n adults, we assume that the probability that an adult survives to the next time step $n+1$ is given by the following survival function:

$$s(A_n) = \frac{a}{1+kA_n}, \quad a \in (0,1) \text{ and } k > 0.$$

The population model is:

$$J_{n+1} = (1-f)J_n + bA_n \tag{1}$$
$$A_{n+1} = fJ_n + s(A_n)A_n \tag{2}$$

Fixed points

Clearly $(J,A) = (0,0)$ is a fixed point of $(1)-(2)$, but are there any others? From (1) we find that at any fixed point there must hold that:

$$A = \frac{f}{b}J \tag{3}$$

and plugging this into (2), we find after rearranging that:

$$(1-b)J = s(\frac{f}{b}J)J. \tag{4}$$

Obviously $J = 0$ solves (4), but assuming that $J \neq 0$, a second, positive solution may exist if the following equation has a positive solution:

$$1-b = s(\frac{f}{b}J). \tag{5}$$

By our assumptions, the right-hand side is a decreasing function of J which tends to 0 as J tends to infinity, while the left-hand side is a positive constant. Therefore, a unique positive solution exists if and only if

$$1-b < s(0) = a.$$

Once this positive solution, which we denote by J^*, is found, we can plug it into (3) to find a positive solution A^*. Thus, we have established that there is a unique positive fixed point (J^*, A^*) if and only if

$$1 < a+b. \tag{6}$$

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In other words, we require that either the survival probability a at low levels of the adult population ($s(0) = a$), or the average number of births per adult b is sufficiently high.

Stability of fixed points

Linearization of $(1) - (2)$ yields the following Jacobian matrix:

$$Jac = \begin{pmatrix} 1-f & b \\ f & \frac{d}{dA}\left(s(A)A\right) \end{pmatrix} = \begin{pmatrix} 1-f & b \\ f & \frac{a}{(1+kA)^2} \end{pmatrix}$$

Evaluating this at the $(0,0)$ fixed point, we find:

$$Jac(0,0) = \begin{pmatrix} 1-f & b \\ f & a \end{pmatrix}, \quad \mathrm{tr}\left(Jac(0,0)\right) = 1 - f + a, \quad \det\left(Jac(0,0)\right) = (1-f)a - fb.$$

By the Jury conditions we know that this fixed point is asymptotically stable if $|\mathrm{tr}J| < 1 + \det J < 2$, and unstable when one of these inequalities is reversed. The second inequality is always satisfied here: $1 + a - f(a+b) < 2$ because $a < 1$ and f, a, and b are positive. Since the trace of J is always positive (because $f < 1$), the first inequality reduces to:

$$1 > a + b \tag{7}$$

Thus, the fixed point $(0,0)$ is asymptotically stable if (7) holds, and unstable if the inequality is reversed, which is precisely condition (6). Recall that (6) is the necessary and sufficient condition for the existence of the positive fixed point. In other words, the fixed point $(0,0)$ is asymptotically stable if and only if it is the only fixed point, and it is unstable if there is a second positive fixed point (J^*, A^*).

Let us investigate stability of (J^*, A^*) by means of the Jury conditions. The second Jury condition at (J^*, A^*) is:

$$\frac{a(1-f)}{(1+kA^*)^2} - fb < 1,$$

which is always satisfied because the first term is less than 1, and fb is positive. The first Jury condition is -after some manipulation:

$$0 < f\left(1 - \frac{a}{(1+kA^*)^2} - b\right).$$

Since $s(A^*) = a/(1 + kA^*)$, and using (3) and (5), the last inequality is satisfied if and only if

$$0 < f\left(1 - a\left(\frac{1-b}{a}\right)^2 - b\right),$$

which in turn is equivalent to

$$0 < \frac{1-b}{a}(a + b - 1)$$

This inequality is satisfied because (6) holds. Thus, whenever the second, positive fixed point (J^*, A^*) exists (and as we know this is the case if and only if (6) holds), it is asymptotically stable.

Problems:

1. What happens to our analysis when we replace the assumption that $b \in (0,1)$ to $b \geq 1$? In other words, what happens when the average number of off-spring per adult is at least one juvenile?

2. What would happen to the adult population if there were no juveniles, i.e. analyze equation (2) without the fJ_{n+1} term.

3. In our model we implicitly assumed that all juveniles survive between two consecutive times. This may not be an appropriate assumption, especially in populations that are exposed to predators. How would you modify model $(1) - (2)$ to the case where juveniles have their own survival function $\bar{s}(J)$ (with a form and interpretation which is similar to the survival function of the adults). As before we still make the assumption that a fraction f *of the surviving juveniles* makes it to adulthood.