Notes on the Jury conditions

Patrick De Leenheer[*]

January 16, 2009

Here we will solve problem 2.4.15 which states that it is necessary and sufficient for a real 2 by 2 matrix J to have eigenvalues with modulus less than 1, that:

$$|\mathrm{tr}J| < 1 + \det J < 2, \tag{1}$$

where $\mathrm{tr}J$ is the trace of J (i.e. the sum of the diagonal entries of J), and $\det J$ is the determinant of J.

The practical relevance of these so-called Jury conditions is that instead of calculating the eigenvalues of J, and checking if their modulus is less than 1, we can verify this with the above inequalities which are expressed in quantities (trace and determinant) that are easily calculated, once J is given.

Proof. Setting

$$J = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix},$$

we see that the characteristic equation $\det(J - \lambda I_2) = 0$ is $\lambda^2 - (j_{11} + j_{22})\lambda + (j_{11}j_{22} - j_{12}j_{21}) = 0$, which in short is

$$p(\lambda) = \lambda^2 - \mathrm{tr}J\lambda + \det J = 0. \tag{2}$$

Let's show that (1) are necessary conditions, but notice first that (1) is the same as the following set of 3 inequalities:

$$-(1 + \det J) < \mathrm{tr}J < 1 + \det J < 2, \tag{3}$$

If the first inequality of (3) does not hold, then $1 + \mathrm{tr}J + \det J \leq 0$, i.e. $p(-1) \leq 0$. Then (2) must have a real root $\lambda^* \leq -1$ (because $p(\lambda) \to +\infty$ as $\lambda \to -\infty$), a contradiction. Similarly, if the second inequality of (3) does not hold, then $p(1) \leq 0$. Then (2) must have a real root $\lambda^* \geq 1$, again a contradiction. Finally, if the third inequality of (3) does not hold, then $\det J \geq 1$, and since $\det J$ is equal to the product of the two roots of (2), it follows that at least one of them has a modulus greater than or equal to 1, a contradiction.

To show sufficiency of the conditions, we will show that if

$$|\lambda_1| \geq 1 \text{ or } |\lambda_2| \geq 1,$$

where λ_1 and λ_2 are roots of (2), then at least one of the inequalities of (3) is violated. Suppose first that λ_1 and λ_2 are complex conjugate roots. Then $\det J = \lambda_1\lambda_2 = |\lambda_1|^2 \geq 1$, violating the third inequality of (3). So from now on we assume that λ_1 and λ_2 are real. Moreover, without loss of generality we assume that $|\lambda_1| \geq 1$ (by relabeling the roots if necessary).

Case 1: $\lambda_1 \geq 1$. If also $\lambda_2 \geq 1$, then $\det J \geq 1$, violating the third inequality of (3). On the other hand, if $\lambda_2 < 1$, then by the intermediate value theorem there must hold that $p(1) \leq 0$, violating the second inequality of (3).

Case 2: $\lambda_1 \leq -1$. The proof is similar as that of Case 1 (complete the steps). □

[*]Email: deleenhe@math.ufl.edu. Department of Mathematics, University of Florida.