Notes on evolutionary game dynamics

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These notes describe some basic concepts from game theory (Nash and strict Nash equilibrium and evolutionary stable strategies), and how these lead to a remarkable equation called the replicator equation which has been used as a model for evolution. They are based largely on [1].

Example: The hawk-dove game.

When animals engage in a conflict (over mates, land ,food etc) they may pick one of two strategies. Either they behave as *hawks* in which case they fight until one of them gets injured or the opponent flees. Or they behave as *doves* in which case they may display aggressive behavior, but they retreat as soon as their opponent shows signs of escalating the conflict into a fight. Assume that the winner of the contest gains $G > 0$ (a mate, some land, some food), and injury leads to a loss of $I > 0$ (no mate, no food, scratches, a decreased level of self-confidence etc). We assume that the cost of an injury is larger than the value of the gain:

$$G - I < 0.$$

Let us consider the outcome of the 4 possible conflict situations:

1. When two hawks meet, there will be a fight with one winner and one loser, and the expected payoff for each hawk is $(G - I)/2$.
2. When a hawk meets a dove, the dove bails out, and the hawk receives $G > 0$.
3. When a dove meets a hawk, the dove flees, and gets nothing, but also does not get harmed, so his payoff is 0.
4. When a dove meets a dove, he might or might not flee, and his expected gain will be $G/2$.

We summarize these outcomes in the following expected **payoff matrix**:

$$A = \begin{pmatrix} \frac{G-I}{2} & G \\ 0 & \frac{G}{2} \end{pmatrix} = \begin{pmatrix} - & ++ \\ 0 & + \end{pmatrix}$$

The first row contains the expected payoffs of a hawk against a hawk and dove respectively. Similarly, the second row contains the payoffs of a dove against a hawk and dove respectively.

Now suppose you are the first player (the row player), and you wish to decide which strategy to pick. Of course, your pick should be such that you maximize your expected payoff. If you knew that your opponent was a hawk strategist, you would obviously pick the dove strategy since $0 > (G - I)/2$. On the other hand, if you knew that your opponent was a dove strategist, you would pick the hawk strategy since $G > G/2$.

But in reality you often don't know your opponent's strategy, and then it is not clear which strategy you should pick.

Moreover, the game might take place repeatedly, and you could learn your opponent's strategy by observing his or her behavior. It would not be wise of your opponent to fix his strategy and stick to it forever after. Smarter would be to behave as a hawk or a dove with a certain probability. Hawk and dove strategies are examples of *pure strategies*. Someone playing hawk 50% of the time and dove 50% is an individual who is playing a *mixed strategy* which can be represented as a probability vector $p = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$. There are many other possible strategies of course. In fact we

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can describe the set of all possible mixed strategies by vectors $\begin{pmatrix} x \\ 1-x \end{pmatrix}$ where $x \in (0,1)$. Pure strategies can be described as well using these vectors by allowing $x = 1$ (for hawk) or $x = 0$ (for dove). They are the unit vectors e_1 and e_2 of the standard basis of $\mathbb{R}^2$.

The wealth of possible strategies to choose from has apparently only complicated matters: both you and your opponent now have an infinite number of strategies to pick from. The question "What is the best strategy to play?" which we could not answer if we restricted to pure strategies alone, does not seem to have become any easier. One of Nash's contributions was to show that there always exist "best strategies" (whatever that means for now). We will first describe a more general framework of a game, and then learn what these optimal strategies really are.

Normal form games We only deal with what are called symmetric games where two players may pick from a fixed number n of pure strategies. The payoff matrix A is a real n by n matrix whose entry in the ith row and jth column contains the payoff for player 1 who played strategy i, when player 2 has played strategy j. Mixed strategies are represented by probability vectors denoted by vectors p having the property that $p_i \in [0,1]$ and $\sum p_i = 1$. The expected payoff for player 1 when he adopts a mixed strategy p and player 2 adopts strategy q is then given by the following dot product of vectors:

$$p.Aq,$$

Indeed, $(Aq)_i$ would be the expected payoff of player 1 playing pure strategy i when player 2 adopts mixed strategy q. Since player 1 plays pure stratey i with probability p_i, the above result follows.

Nash equilibrium Assume player 2 adopts strategy q. Then player 1 who is trying to maximize his payoff should pick a strategy p which is such that the linear function $p \to f(p) = p.Aq$ is maximal. By continuity, and by compactness of the set that contains all probability vectors, we know that such a strategy will always exist. In fact, there may be several strategies p that maximize $p.Aq$. For a given strategy q, we denote the set containing all these maximizers by $B(q)$, and we call this set the set of best replies to strategy q.

Definition 1. *A strategy q is called a Nash equilibrium if $q \in B(q)$. In other words, q is a Nash equilibrium if it is a best reply against itself:*

$$p.Aq \leq q.Aq, \text{ for all probability vectors } p. \tag{1}$$

We see that player 1 cannot do better than to play strategy q if player 2 plays the Nash equilibrium strategy q. What John Nash did in his 1950 PhD thesis [1] was show that **every game has at least one Nash equilibrium**. [For math students: the proof is based on an application of a fixed point theorem by Kakutani [2]]

A stronger notion than that of a Nash equilibrium is the following:

Definition 2. *A strategy q is called a strict Nash equilibrium if $B(q) = \{q\}$. In other words, q is a strict Nash equilibrium if it is* **the only** *best reply against itself:*

$$p.Aq < q.Aq, \text{ for all probability vectors } p \neq q. \tag{2}$$

Caution: Unlike Nash equilibria, strict Nash equilibria do not necessarily exist in every game. For example, we will see shortly that the hawk-dove game does not have strict Nash equilibria.

Let us now determine the Nash equilibria for all games with 2 pure strategies, and show which ones are strict. We denote the payoff matrix by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are arbitrary real numbers, and the strategy vectors are written as:

$$p = \begin{pmatrix} x \\ 1-x \end{pmatrix}, \quad x \in [0,1] \text{ and } q = \begin{pmatrix} y \\ 1-y \end{pmatrix}, \quad y \in [0,1].$$

[1] http://en.wikipedia.org/wiki/John_Forbes_Nash A sketch of the proof is in a link to Nash's 1 page paper *Equilibrium points in N-person games*, PNAS 36, 48-49, 1950.

[2] http://en.wikipedia.org/wiki/Kakutani_fixed_point_theorem

Then, setting $\alpha = a - c$ and $\beta = b - d$, and suppressing some tedious algebra, (1) amounts to:

$$0 \le (y - x)(\alpha y + \beta(1 - y)) \text{ for all } x \in [0, 1]. \tag{3}$$

and (2) to:

$$0 < (y - x)(\alpha y + \beta(1 - y)) \text{ for all } x \in [0, 1] \text{ with } x \neq y. \tag{4}$$

We will split up the discussion in several cases, depending on the signs of α and β.

1. $\alpha = \beta = 0$. In this case (3) is always satisfied, but (4) is not. *In this case, all strategies q are Nash equilibria, but none of them are strict.*

2. $\alpha = 0$, $\beta > 0$. Then (3) is satisfied if and only if $y = 1$. Indeed, if $y < 1$, then the factor $y - x$ can always be made negative by a suitable choice of x, and then inequality (3) would fail. This means that $q = e_1$ (the first pure strategy) is a Nash equilibrium, but since (4) fails it is not strict. *In this case, the first pure strategy e_1 is the only Nash equilibrium, but it is not strict.*

3. $\alpha = 0$, $\beta < 0$. A similar reasoning shows that *in this case, the first pure strategy e_1 is a non-strict Nash equilibrium, and the second pure strategy e_2 is a strict Nash equilibrium. There are no other Nash equilibria.* (**HW**)

4. $\alpha > 0$, $\beta = 0$. *In this case, the first pure strategy e_1 is a strict Nash equilibrium, and the second pure strategy e_2 is a non-strict Nash equilibrium. There are no other Nash equilibria.* (verify this)

5. $\alpha < 0$, $\beta = 0$. *In this case, the second pure strategy e_2 is the only Nash equilibrium, and it is not strict.* (verify this)

6. $\alpha, \beta > 0$. Note that $\alpha y + \beta(1 - y)$ is a convex combination of the positive numbers α and β, hence it is necessarily positive. Consequently, *in this case, the first pure strategy e_1 is the only Nash equilibrium, and it is strict.*

7. $\alpha, \beta < 0$. *In this case, the second pure strategy e_2 is the only Nash equilibrium, and it is strict.* (**HW**)

8. $\alpha < 0$, $\beta > 0$. It is not hard to see that the pure strategies e_1 ($y = 1$) and e_2 ($y = 0$) are not Nash equilibria. The factor $(\alpha y + \beta(1 - y))$ in (3) is zero if $y = \frac{\beta}{\beta - \alpha} \in (0, 1)$, hence *in this case there is a single, non-strict Nash equilibrium at the mixed strategy:*

$$q = \begin{pmatrix} \frac{\beta}{\beta-\alpha} \\ \frac{-\alpha}{\beta-\alpha} \end{pmatrix} \tag{5}$$

9. $\alpha > 0$, $\beta < 0$. *In this case the pure strategies e_1 ($y = 1$) and e_2 ($y = 0$) are strict Nash equilibria, and the mixed strategy* (5) *is a non-strict Nash equilibrium.* (**HW**)

All these cases are summarized in Figure 1.

This case study shows that although there are always Nash equilibria, these are not necessarily unique (cases 8 and 9). Strict Nash equilibria do not always exist (cases 1,2,5 and 8).

Illustration on the hawk-dove game This game is an example of case 8: $\alpha = (G - I)/2 - 0 < 0$ and $\beta = G - G/2 = G/2 > 0$. There is a unique, non-strict Nash equilibrium at the mixed strategy:

$$q = \begin{pmatrix} \frac{G}{I} \\ \frac{I-G}{I} \end{pmatrix}$$

Notice that at the Nash equilibrium q, the larger the quantity $I - G$ is (i.e. the larger the difference between the cost of an injury and the gain of a contest), the larger the probability to adopt the dove strategy tends to be. This agrees with the observation that the dove strategy is widespread in heavily armed animals (where I is very large), see [2] (there is a link to this paper on the course webpage).

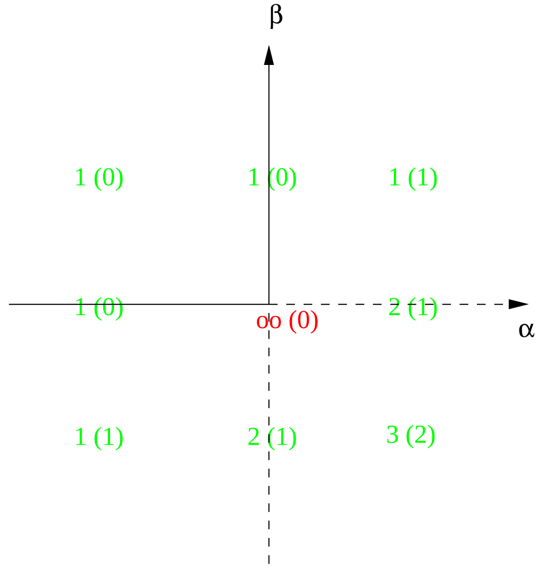

Figure 1: $\alpha = a - c$ and $\beta = b - d$. The number of Nash equilibria is given in each region, and followed by the number of strict Nash equilibria in ().

Evolutionary stable strategies (ESS) We have seen that strict Nash equilibria are strategies that are the only best replies against themselves. If you suspect that your opponent is a strict Nash strategist, then you should not deviate from that strategy yourself, as your payoff would be lower. Unfortunately, as we have seen, strict Nash equilibria do not always exist. What strategy to choose in this case??

We have also seen that Nash equilibria always exist. They are strategies that form a best reply against themselves. But if they are not strict, there might be an alternative best reply, and adopting this strategy yields the same payoff. When is it smart to adopt this alternative strategy, and when is it not? We will see that this leads to a refinement of the concept of a Nash equilibrium to that of an evolutionary stable strategy, which is due to the evolutionary biologist John Maynard Smith[3].

Definition 3. *A strategy q is an ESS if:*

1. *$p.Aq \leq q.Aq$ for all probability vectors p. (**Nash equilibrium condition**)*
2. *If $p'.Aq = q.Aq$ for some $p' \neq q$, then $q.Ap' > p'Ap'$. (**stability condition**)*

Note that the first condition says that q should be a Nash equilibrium. Every ESS is thus a Nash equilibrium. The second condition implies that if there is an alternative best reply p' to q, then p' is NOT a best reply against itself (as q yields a higher payoff), i.e. p' is not a Nash equilibrium. The following implications are immediate from the definitions:

$$q \text{ is strict Nash} \Rightarrow q \text{ is ESS} \Rightarrow q \text{ is Nash.}$$

The way to find all ESS of a game is as follows:

1. Find all Nash equilibria q.
2. Fix a Nash equilibrium q, and determine its corresponding set of best replies $B(q)$. For each $p' \in B(q)$ with $p' \neq q$, the stability condition should hold.

Caution: Unlike Nash equilibria, ESS may not exist for a given game. This will become clear when we discuss the rock-scissors-paper game.

Illustration on the hawk-dove game We have already determined that this game has only one (non-strict) Nash equilibrium

$$q = \begin{pmatrix} \frac{G}{I} \\ \frac{I-G}{I} \end{pmatrix}.$$

[3]http://en.wikipedia.org/wiki/John_Maynard_Smith

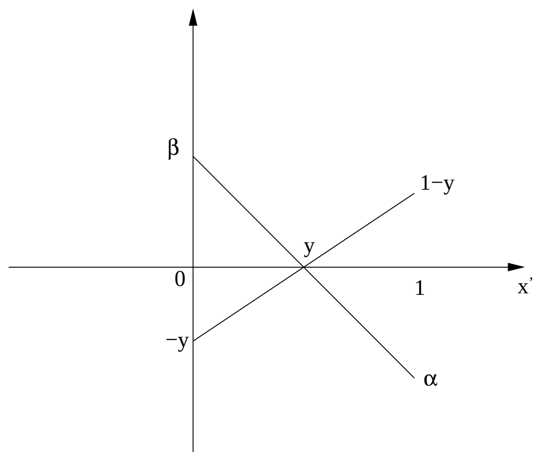

Figure 2: Graphs of the functions $(y - x')$ and $(\alpha x' + \beta(1 - x'))$ appearing in (6). $\alpha = (G - I)/2 < 0$, $\beta = G/2 > 0$ and $y = G/I \in (0, 1)$.

What is the set of best replies $B(q)$? Recall that for the hawk-dove game, the second factor in (3) is zero, and thus *every mixed strategy p' is a best reply to q*! To see if the stability condition holds, we need to check if for all $x' \in [0, 1]$ with $x' \neq y$:

$$0 > (x' - y)(\alpha x' + \beta(1 - x')) \tag{6}$$

holds.[4]

In this example, $y = G/I$ and $\alpha = (G - I/2 < 0)$ and $\beta = G/2 > 0$. The right hand side in (6) is the product of two linear functions in x' on the interval $[0, 1]$. The first factor is increasing with x' and it crosses the x'-axis at y. The second factor is decreasing in x' (because $\alpha - \beta < 0$) and it crosses the x'-axis at the same point y. This is shown in Figure 2. Thus, the product is indeed negative if $x' \neq y$, and hence (6) is satisfied. Therefore, q is indeed an ESS.

The replicator equation Each individual in a large population might adopt a pure strategy of some underlying game. Individuals can engage in a game by randomly choosing an opponent. Depending on the outcome of the game, the individuals might decide to modify their strategy in the next round, and so the number of individuals adopting a certain pure strategy will usually fluctuate in time. What drives these fluctuations? According to the theory of evolution, successful strategists (those with higher payoffs) should be selected for. Can this be described by a mathematical model? We will study one such model, the replicator equation which captures some interesting features of the process of evolution.

Consider a game with n pure strategies and the n by n payoff matrix A. Let x_i be the fraction of strategists in a large population that use the pure strategy i. We define the vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

whose components belong to $[0, 1]$ and add up to 1. It is called the state vector of the population. We wish to derive a differential equation for each fraction x_i. But what governs the rate of change of x_i? Imagine you are an e_i strategist, and suppose that the state of the population is given by the vector x. If you randomly pick an opponent from the population, your expected payoff will be:

$$e_i.Ax = (Ax)_i$$

This expected payoff should be compared to the expected payoff of two randomly chosen individuals, which is:

$$x.Ax$$

[4]Notice that (6) is obtained from (3) by reversing the inequality in (3) and making it strict, and by replacing x by y and y by x'. That this is correct follows from comparing the structure of the conditions in the definition of an ESS.

We postulate that the per capita rate of change of the fraction of e_i strategists, is proportional to x_i with proportionality factor equal to the difference of both expected payoffs:

$$(Ax)_i - x.Ax.$$

This factor is positive if $(Ax)_i - x.Ax > 0$, and hence the fraction of e_i strategists will increase. This is in agreement with the evolutionary principle that successful strategists should thrive. If, on the other hand, $(Ax)_i - x.Ax < 0$, then the fraction of e_i strategists should decrease. Summarizing, we have just introduced the replicator equation:

$$\dot{x}_i = x_i((Ax)_i - x.Ax), \quad i = 1, \dots, n \tag{7}$$

This can be written more compactly in matrix notation:

$$\dot{x} = \mathrm{diag}(x)(Ax - x.Ax\mathbf{1}), \tag{8}$$

where $\mathbf{1}$ denotes a column vector whose entries are all equal to 1.

Some properties of the replicator equation The state vector x of the replicator equation is a probability vector. This means that the solutions $x(t)$ of (7) should always be such that if the components of $x(0)$ add up to 1, then the same must be true for the components of $x(t)$ for all t. Let us verify this by calculating:

$$\begin{aligned} \frac{d}{dt}\left(\sum_{i=1}^{n} x_i(t)\right) &= \sum_{i=1}^{n} \dot{x}_i(t) = \sum_{i=1}^{n} x_i(t)(Ax(t))_i - (x(t).Ax(t))(\sum_{i=1}^{n} x_i(t)) \\ &= x(t).Ax(t) - (x(t).Ax(t)).1 = 0, \text{ for all t}, \end{aligned}$$

as required.

Similarly, if $x_i(0) \geq 0$ then $x_i(t) \geq 0$ for all t. This can be checked by first noticing the following fact:

$$\text{If } x_i(0) = 0, \text{ then } x_i(t) = 0, \text{ for all t},$$

by uniqueness of solutions of the system of differential equations (7). Consequently, if $x_i(0) \geq 0$, then $x_i(t) \geq 0$ for all t, for if this were not the case, there would be some time t^* where $x_i(t^*) < 0$. Since $x_i(t)$ is a continuous function of t, there would be some t' where $x_i(t') = 0$. In other words, there is some solution with the property that $x_i(t') = 0$ and $x_i(t^*) < 0$, contradicting the above fact.

A concise way to summarize all this is by saying that the *unit simplex*

$$S_n = \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \text{ and } \sum_{i=1}^{n} x_i = 1\}$$

is an *invariant set* [5] for system (8).

We show the unit simplex in Figure 3 in case $n = 2$ and 3.

Relationship between the replicator equation and the underlying game What is the relation between the various notions of the underlying game (Nash equilibrium, ESS), and the corresponding replicator equation?

Theorem 1. *1. Every pure strategy e_j of the underlying game is a steady state of the replicator equation* (8).

2. Let $\bar{x}$ be a Nash equilibrium of the underlying game. Then $\bar{x}$ is a steady state of the replicator equation (8).

Proof. 1. Let e_j be a pure strategy of the game. We need to check that for $x = e_j$, there holds that:

$$x_i((Ax)_i - x.Ax) = 0, \text{ for all } j = 1, \dots, n$$

This is obvious for all $i \neq j$ because then $x_i = (e_j)_i = 0$. It is also obvious for $i = j$ because then $(Ae_i)_i - e_i.Ae_i = (Ae_i)_i - (Ae_i)_i = 0$.

[5] In general, a set A is an invariant set of a system of ODE's $\dot{x} = f(x)$, if solutions that start in that set, remain in it. More precisely, if $x(0) \in A$, then $x(t) \in A$ for all t.

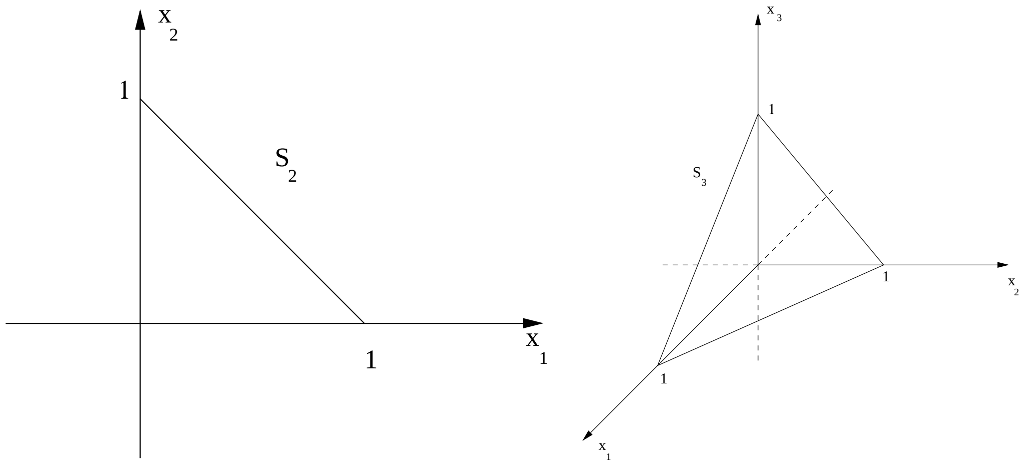

Figure 3: The unit simplices for the case $n = 2$ and $n = 3$.

2. Let $\bar{x}$ be a Nash equilibrium of the underlying game. Then

$$y.A\bar{x} \leq \bar{x}.A\bar{x}, \text{ for all } y \in S_n$$

This is true in particular when $y = e_i$, and thus

$$e_i.A\bar{x} \leq \bar{x}.A\bar{x} \tag{9}$$

We claim that this inequality is actually an equality for all i where $\bar{x}_i > 0$:

$$e_i.A\bar{x} = \bar{x}.A\bar{x}, \text{ for all } i \text{ such that } \bar{x}_i > 0. \tag{10}$$

Suppose not, then there would be some j with $\bar{x}_j > 0$, such that $e_j.A\bar{x} < \bar{x}.A\bar{x}$. For each $\bar{x}_i > 0$, we can multiply (9) by $\bar{x}_i$, and add over all i. Since a term corresponding to j is included in this sum, the result is that:

$$\sum_{i:\, \bar{x}_i > 0} \bar{x}_i e_i.A\bar{x} = \bar{x}.A\bar{x} < \sum_{i:\, \bar{x}_i > 0} \bar{x}_i(\bar{x}.A\bar{x}) = 1.(\bar{x}.A\bar{x}) = \bar{x}.A\bar{x},$$

a contradiction. That $\bar{x}$ is a steady state of the replicator equation (8) is now not hard to see. If $\bar{x}_i = 0$, then clearly $\bar{x}_i((A\bar{x})_i - \bar{x}.A\bar{x}) = 0$, while if $\bar{x}_i > 0$, this is still the case by (10). □

Caution: Not every steady state of the replicator equation (8) is a Nash equilibrium of the underlying game. For instance, the two pure strategies in the hawk-dove game are steady states of the replicator equation by the first part of the Theorem, yet, as we have seen earlier, they are not Nash equilibria of the underlying game.

One of the most important results regarding the replicator equation involves the notion of an ESS and is stated next without proof, which can be found in [1].

Theorem 2. *Let $\bar{x}$ be an ESS of the underlying game. Then $\bar{x}$ is an asymptotically stable steady state of* (8)*. Moreover, if $\bar{x}$ belongs to the interior of the unit simplex S_n, then all solutions starting in the interior of S_n converge to $\bar{x}$.*

Illustration on the hawk-dove game We have already established that the mixed strategy

$$\begin{pmatrix} \frac{G}{I} \\ \frac{I-G}{I} \end{pmatrix}$$

is an ESS, and that it is the only Nash equilibrium of the underlying game.

The corresponding replicator equation has 3 steady states: e_1, e_2 (by Theorem 1) and also

$$\bar{x} = \begin{pmatrix} \frac{G}{I} \\ \frac{I-G}{I} \end{pmatrix}$$

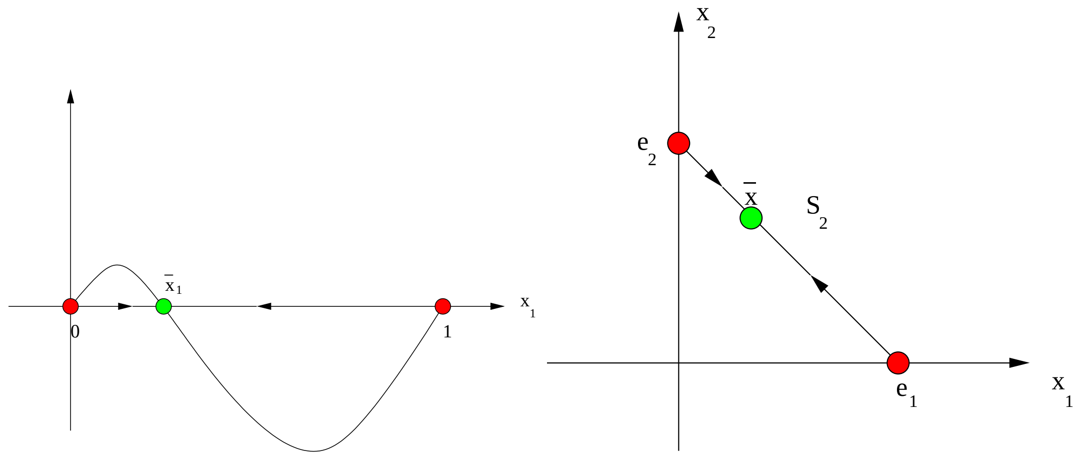

Figure 4: Phase-line analysis for (11) (left panel), and for (8) (right panel).

by Theorem 2. In fact, by the same result, $\bar{x}$ is asympotically stable, and all solutions in the interior of S_2 converge to it. Let us verify this by performing an independent phase line analysis. Since $n = 2$, and since we know that for the replicator equation (8), there holds that $x_1(t) + x_2(t) = 1$, it is enough to analyze the dynamics of the scalar ODE for x_1:

$$\begin{aligned}\dot{x}_1 &= x_1((Ax)_1 - x.Ax) = x_1\left((Ax)_1 - x_1(Ax)_1 - (1-x_1)(Ax)_2\right) = x_1(1-x_1)((Ax)_1 - (Ax)_2) \\ &= x_1(1-x_1)(\alpha x_1 + \beta(1-x_1)),\end{aligned}$$

where $\alpha = a - c$ and $\beta = b - d$ as before. This can be simplified to:

$$\dot{x}_1 = x_1(1-x_1)\left(\frac{G-I}{2}x_1 + \frac{G}{2}(1-x_1)\right), \tag{11}$$

using the specific payoff matrix A of the hawk-dove game.

The product of the first two factors on the right hand side is positive in $(0,1)$ and 0 in $x_1 = 0$ and $x_1 = 1$. The third factor is a linear function in x_1, which has a zero when $x_1 = G/I$, and it is equal to $a - c = G/2 > 0$ when $x_1 = 0$. The phase-line analysis for equation (11) is in the left panel of Figure 4, and it confirms that the steady state at $x_1 = G/I$ is asymptotically stable. The two steady state $x_1 = 0$ and $x_1 = 1$ are clearly unstable. This analysis implies that the steady state $\bar{x}$ is asymptotically stable for (8), and that the equilibria e_1 and e_2 are unstable, see the right panel of Figure 4.

The rock-scissors-paper game You are probably familiar with this game. We will see that the replicator equation associated to a very special case of this game yields some interesting dynamical behavior. Recall that rock beats scissors, that scissors beats paper, and paper beats rock. We assume that the 3 by 3 payoff matrix is (R is 1st pure strategy, S is 2nd and P is 3rd)

$$A = \begin{pmatrix} 0 & +1 & -1 \\ -1 & 0 & +1 \\ +1 & -1 & 0 \end{pmatrix}$$

We know that the system dynamics for (8) takes place on the unit simplex S_3, depicted in the right panel of Figure 3. Let us start by calculating steady states. By Theorem 1, the 3 corners e_1, e_2 and e_3 of S_3 are steady states. Are there any other steady states on the boundary of S_3 (the boundary consists of all those states where at least one component of the state vector x is 0). For instance, are there steady states with $x_3 = 0$? If there are, they must satisfy that x_1 and x_2 are positive with $x_1 + x_2 = 1$, and:

$$(Ax)_1 = (Ax)_2 = x.Ax$$

It is easily verified that $x.Ax = 0$ no matter what x is, and thus these equations become:

$$x_2 = -x_1 = 0 \text{ and } x_1, x_2 > 0 \text{ with } x_1 + x_2 = 1,$$

which is impossible. Hence, there are no other steady states on the boundary of S_3 where $x_3 = 0$. Similarly, one can show that there are no other steady states on the boundary of S_3 where $x_2 = 0$ or where $x_1 = 0$. Finally, are there steady states in the interior of S_3. If so, then these must be such that $x_1, x_2, x_3 > 0$ with $x_1 + x_2 + x_3 = 1$ and:

$$Ax = 0,$$

i.e.

$$x_2 - x_3 = -x_1 + x_3 = x_1 - x_2 = 0.$$

Clearly, there is a unique solution:

$$x^* = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

Let us now study the dynamical behavior of (8). First notice that the boundary faces of S_3 are invariant sets.[6] To see this, notice that if $x_3(0) = 0$, then $dx_3/dt = 0$, and hence $x_3(t) = 0$ for all t. In other words, if a solution starts in F_3, then it remains there. (Similar arguments hold for F_1 and F_2) What happens to solutions starting in F_3? They are given by the following two equations:

$$\begin{aligned} \dot{x}_1 &= x_1(x_2) \\ \dot{x}_2 &= x_2(-x_1), \end{aligned}$$

where $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S_2$. This is again a replicator equation! It has exactly two steady states corresponding to two corners, and all solutions converge to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This simply says that if only R and S were possible strategies to be played, then ultimately all individuals in the population would play R, as expected. Similar arguments reveal the dynamics on the other invariant sets F_1 and F_2. We illustrate the resulting dynamical behavior of the solutions of (8) that lie on the boundary of S_3 in Figure 5. We have just constructed an example of a *heteroclinic cycle*. It consists of 3 steady states e_1, e_2 and e_3 joined by 3 solutions that connect e_1 to e_2, e_2 to e_3 and e_3 to e_1.

What happens to solutions that start in the interior of S_3? It turns out that that the system has a *conserved quantity* (this is a quantity which does not change along solutions of the system). Consider the following function:

$$P(x) = x_1 x_2 x_3,$$

[6] The 3 boundary faces of S_3 are the sets $F_1 = \{x \in S_3 | x_1 = 0\}$, $F_2 = \{x \in S_3 | x_2 = 0\}$ and $F_3 = \{x \in S_3 | x_3 = 0\}$.

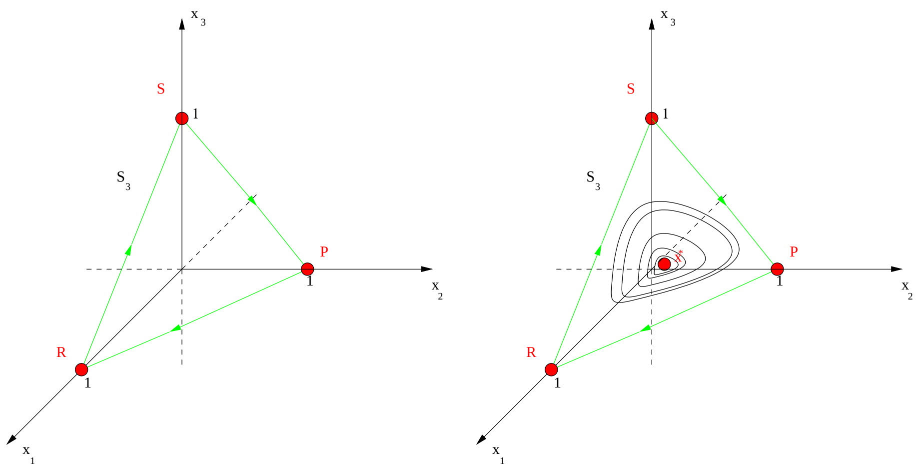

Figure 5: Left panel: The heteroclinic cycle of replicator equation corresponding to the rock-scissors-paper game. Right panel: Periodic solutions for the replicator equation corresponding to the rock-scissors-paper game. (solutions are traversed clockwise)

and let us see what happens to the value of P along an arbitrary solution $x(t)$ of system (8). To that end, we calculate:

$$\begin{aligned}
\dot{P} &= \frac{d}{dt}\left(P(x(t))\right) \\
&= \dot{x}_1 x_2 x_3 + x_1 \dot{x}_2 x_3 + x_1 x_2 \dot{x}_3 \\
&= x_1 x_2 x_3 \left((Ax)_1 + (Ax)_1 + (Ax)_3 - 3x.Ax\right) \\
&= x_1 x_2 x_3 \left((x_2 - x_3) + (-x_1 + x_3) + (x_1 - x_2) + 0\right) \\
&= 0.
\end{aligned}$$

In other words, the value of P does not change along any solution of (8), and thus P is a conserved quantity.

Now imagine what the level set of the function P look like (recall that a level set of the function P is defined as the set of points x where $P(x)$ equals a given constant). They look like cup shaped surfaces in $\mathbb{R}^3$ that have hyperbolic intersections with planes parallel to the coordinate planes. Since solutions are confined to move on such surfaces, and also confined to the unit simplex S_3, it is not hard to see that the solution curves in the interior of S_3 are in fact closed curves, see Figure 5. These closed curves therefore correspond to periodic solutions of system (8). Notice that the steady state x^* corresponds to a particular periodic solution, namely to a closed curve which has degenerated into a point. The steady state x^* is surrounded by a band of infinitely many periodic solutions that accumulate into the heteroclinic cycle on the boundary of S_3.

These results tell us that if we would observe a large population of rock-scissors-paper players, we will see periodic fluctuations in the fractions of players that use a certain strategy.

HW problems:

1. **The prisoner's dilemma**. In this game two people are suspected of committing a serious crime, but the prosecutor has not enough proof for a conviction. However, they have also committed a minor offense, and the prosecutor's case for this crime is a slam dunk. Of course, what he wants is a confession of the serious crime. So he puts the suspects in separate interrogation rooms and questions them simultaneously, without the possibility of communication between them. If both suspects deny, then suspect 1 gets to do 1 year of prison time for the minor offense. If suspect 1 denies but suspect 2 confesses, then suspect 1 gets $1 + 9 = 10$ years of prison time for both crimes.. for both crimes. If suspect 1 confesses and suspect 2 denies, then suspect 1 is set free. If both confess, then suspect 1 gets 9 years of prison time for the serious crime. If the first pure strategy is to deny, and the second to confess, then the payoff matrix is

$$A = \begin{pmatrix} -1 & -10 \\ 0 & -9 \end{pmatrix} = \begin{pmatrix} - & -- \\ 0 & - \end{pmatrix}$$

   If only pure strategies are played, then no matter which pure strategy suspect 2 chooses, suspect 1 should choose to confess ($0 > -1$ and $-9 > -10$). But since the situation is the same for suspect 2 by symmetry of the game, this means that both will end up doing 9 years. The dilemma lies in the fact that if both suspects would have denied, then they would only have gotten 1 year!

   Determine Nash equilibria, strict Nash equilibria and ESS. Analyze the corresponding replicator equation.

2. **The thug-gentleman game**. In this game we consider the behavior of men engaging in conflicts. As in the hawk-dove game, we denote the gain to the winner of a contest by $G > 0$, and the loss by $I > 0$, but now we assume that

$$G - I > 0.$$

   In other words, the cost of an injury is not very high.

   There are two strategies, namely the thug and gentleman strategy. Gentleman strategists really don't care about the gain and they will decline it, even if they win a fight (noble as they are...). Gentlemen are also stronger than thugs and will always defeat them. So when a gentleman meets a gentleman, his expected payoff is $(0 - I)/2 = -I/2$ because he declines the gain in case he wins. When a gentleman meets a thug, he beats the thug but declines the gain, and so his expected payoff is 0. When a thug meets a gentleman, he is beaten, and so the expected payoff of the thug is $-I$. When two thugs meet, their expected payoff is $(G - I)/2$. We have the following payoff matrix (thug is the first pure strategy):

$$A = \begin{pmatrix} \frac{G-I}{2} & -I \\ 0 & -\frac{I}{2} \end{pmatrix} = \begin{pmatrix} + & -- \\ 0 & - \end{pmatrix}$$

   Determine Nash equilibria, strict Nash equilibria and ESS. Analyze the corresponding replicator equation.

References

[1] J. Hofbauer and K. Sigmund, Evolutionary games and replicator dynamics, Cambridge University Press, 1998.

[2] J. Maynard Smith and G.R. Price, The logic of animal conflict, Nature 246, p. 15-18, 1973.