Notes on the diffusion equation

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In previous notes we considered the diffusion equation on $\mathbb{R}$, and obtained the fundamental solution. Here we study the equation on a finite interval $[0, l]$ under two types of boundary conditions, namely Dirichlet and Neumann boundary conditions.

Dirichlet Let

$$u_t = Du_{xx}, \quad u(0,t) = u(l,t) = 0 \text{ for all } t > 0. \tag{1}$$

We will not specify an initial condition, and only attempt to find nonzero solutions with a particular form using the method of **separation of variables**:

$$u(x,t) = X(x)T(t).$$

Plugging this into the diffusion equation we find that

$$\frac{T'}{DT} = \frac{X''}{X} = -\lambda \tag{2}$$

for some constant λ which is yet to be determined. Then $T(t) = T(0)\,\mathrm{e}^{-D\lambda t}$, and since we are not interested in trivial solutions we assume that $T(0) \neq 0$. Then $u(0,t) = u(l,t) = 0$ implies that

$$X(0) = X(l) = 0.$$

This leads to the **boundary value problem (BVP)**:

$$X'' + \lambda X = 0, \quad X(0) = X(l) = 0.$$

There are 3 cases to consider: $\lambda < 0, = 0, > 0$.

Case 1: $\lambda < 0$. By standard results of linear 2nd order ODE's we first find the general solution:

$$X(x) = c_1\,\mathrm{e}^{\sqrt{-\lambda}x} + c_2\,\mathrm{e}^{-\sqrt{-\lambda}x}.$$

The BC $X(0) = X(l) = 0$ then imply that $c_1 = c_2 = 0$, so we find a trivial solution which must be discarded. In other words, λ cannot be negative.

Case 2: $\lambda = 0$. A similar analysis also leads to a trivial solution, and thus λ cannot be zero. (check this as a weekly **HW**)

Case 3: $\lambda > 0$. Solving the ODE we find:

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

The BC $X(0) = X(l) = 0$ implies that:

$$c_1 = 0 \text{ and } c_2 \sin(\sqrt{\lambda}l) = 0.$$

To find a nontrivial solution $X(x)$ we need that $c_2 \neq 0$, and this is only possible if we choose:

$$\sin(\sqrt{\lambda}l) = 0,$$

or equivalently:

$$\lambda_n = (n\pi/l)^2, \quad n = 1, 2, \ldots$$

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Summarizing, using the method of separation of variables we have found an infinite family of solutions with a special form:

$$u_n(x,t) = C_n \, \mathrm{e}^{-D(n\pi/l)^2 t} \sin(n\pi x/l), \quad n = 1, 2, \ldots,$$

where C_n is an arbitrary nonzero constant.

Remark 1: We see that for all $n = 1, 2, \ldots$ and all $x \in (0, l)$:

$$\lim_{t \to \infty} u_n(x,t) = 0,$$

which is similar to what we learned about the fundamental solution when studying the diffusion equation on $\mathbb{R}$.

Remark 2 Although we did not specify an initial condition $u(x,0) = u_0(x)$, we see that we could have done so: In the special case that the initial condition is:

$$u_0(x) = C \sin(m\pi x/l) \text{ for some } m,$$

the solution that satisfies both (1) and the IC is:

$$u(x,t) = C \, \mathrm{e}^{-D(m\pi/l)^2 t} \sin(m\pi x/l).$$

By linearity of the equation, this can be extended to IC's of the following form:

$$u_0(x) = \sum_{j=1}^{m} C_j \sin(j\pi x/l) \text{ for some } m,$$

yielding the solution

$$u(x,t) = \sum_{j=1}^{m} C_j \, \mathrm{e}^{-D(j\pi/l)^2 t} \sin(j\pi x/l)$$

But what about a general initial condition:

$$u(x,0) = u_0(x),$$

where $u_0(x)$ does not have any of the particular forms above? This situation will not be discussed here because it requires the development of Fourier series, a topic outside the scope of this course. Nonetheless, some of the ideas are briefly sketched next. It turns out that under rather weak conditions on $u_0(x)$, we can write $u_0(x)$ as a so-called Fourier series:

$$u_0(x) = \sum_{j=1}^{\infty} C_j \sin(j\pi x/l),$$

where the so-called Fourier coefficients C_j can be calculated based on the function u_0. The next step is then to show that the solution of (1) with this IC is the following Fourier series:

$$u(x,t) = \sum_{j=1}^{\infty} C_j \, \mathrm{e}^{-D(j\pi/l)^2 t} \sin(j\pi x/l).$$

Note the similarity between this formula and the one above for the particular IC, which is in fact just a truncated Fourier series.

Neumann Let

$$u_t = Du_{xx}, \;\; u_x(0,t) = u_x(l,t) = 0 \text{ for all } t > 0. \tag{3}$$

As before we don't specify an initial condition, and we look for solutions using separation of variables:

$$u(x,t) = X(x)T(t)$$

This leads to (2), and then in the same way to the following BVP

$$X'' + \lambda X = 0, \;\; X'(0) = X'(l) = 0.$$

Again we consider 3 cases depending on the sign of λ.

Case 1: $\lambda < 0$. First we find the general solution:

$$X(x) = c_1 \, \mathrm{e}^{\sqrt{-\lambda}x} + c_2 \, \mathrm{e}^{-\sqrt{-\lambda}x}.$$

But since the BC are stated for the derivatives of X at $x = 0, 1$, we calculate:

$$X'(x) = \sqrt{-\lambda}\left(c_1 \, \mathrm{e}^{\sqrt{-\lambda}x} - c_2 \, \mathrm{e}^{-\sqrt{-\lambda}x}\right).$$

Then the BC $X'(0) = X'(l) = 0$ imply that $c_1 = c_2 = 0$. (**verify this**)

Case 2: $\lambda = 0$. In this case the general solution is:

$$X(x) = c_1 x + c_2,$$

and both BC lead to the same condition $c_1 = 0$. Thus, for any $c_2 \neq 0$, we have that $X(x) = c_2$ solves the BVP. Together with $T(t) = T(0)\,\mathrm{e}^{0t}$, this yields the following nonzero solution to (3):

$$u_0(x.t) = C_0, \;\; C_0 \neq 0.$$

Case 3: $\lambda > 0$. Solving the ODE we find:

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \Rightarrow X'(x) = \sqrt{\lambda}\left(-c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x)\right)$$

The BC $X'(0) = X'(l) = 0$ imply that:

$$c_2 = 0 \text{ and } c_1 \sin(\sqrt{\lambda}l) = 0.$$

To find a nontrivial solution $X(x)$ we need that $c_1 \neq 0$, and this is only possible if we choose:

$$\sin(\sqrt{\lambda}l) = 0,$$

or equivalently:

$$\lambda_n = (n\pi/l)^2, \;\; n = 1, 2, \ldots$$

We can combine the results from cases 2 and 3 into a single formula that describes an infinite family of solutions of (3):

$$u_n(x,t) = C_n \, \mathrm{e}^{-D(n\pi/l)^2 t} \cos(n\pi x/l), \;\; n = 0, 1, 2, \ldots,$$

where C_n is an arbitrary nonzero constant. Compared to the Dirichlet case we have $\cos(n\pi x/l)$ instead of $\sin(n\pi x/l)$. The reason for this difference is that the latter is zero in the boundary points $x = 0, 1$, while *the derivative* of the former is zero there. A more important difference is that the Neumann case yields a nonzero constant solution $u_0(x,t) = C_0$, while there are no nonzero constant solutions in the Dirichlet case.

Remark 3 Just as in the Dirichlet case, we have that for all $n = 1, 2, \ldots$ and $x \in (0, l)$:

$$\lim_{t\to\infty} u_n(x,t) = 0,$$

but for $n = 0$ we have that

$$\lim_{t\to\infty} u_0(x,t) = C_0 \neq 0$$

Remark 4 Similar remarks we made in the Dirichlet case about IC's can be made here.