Notes on the fundamental solution of the diffusion equation

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On p. 105 of our text it is claimed that the diffusion equation on $\mathbb{R}$ with the Dirac delta function as initial condition:

$$u_t = Du_{xx}, \quad u(x,0) = \delta_0(x)$$

has the following solution:

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}}\, \mathrm{e}^{-\frac{x^2}{4Dt}},$$

also called the *fundamental solution*.

It is not shown how this solution is obtained, and these notes will outline a way to do it.

Dilation Let $u(x,t)$ be any solution of $u_t = u_{xx}$. Then given a parameter $m \neq 0$, we see that $u(mx, m^2 t)$ is also a solution (just plug in the latter function in the equation and see that it satisfies it, regardless of the value of m!). This suggests we might attempt to find solutions that depend on the ratio x^2/t instead of on the pair (x,t). Therefore, we let $u(x,t)$ be of the following form:

$$u(x,t) = v\left(\frac{x^2}{t}\right),$$

for some appropriate function v, yet to be determined. In this case,

$$u_t = -\frac{x^2}{t^2} v'\left(\frac{x^2}{t}\right) \text{ and } u_x = \frac{2x}{t} v'\left(\frac{x^2}{t}\right) \text{ and } u_{xx} = \frac{2}{t} v'\left(\frac{x^2}{t}\right) + \frac{4x^2}{t^2} v''\left(\frac{x^2}{t}\right),$$

and the function v should satisfy the following ODE:

$$D\frac{4x^2}{t^2} v'' + \left(D\frac{2}{t} + \frac{x^2}{t^2}\right) v' = 0,$$

or with $y = x^2/t$:

$$v''(y) + \left(\frac{1}{2y} + \frac{1}{4D}\right) v'(y) = 0$$

Integrating once we find that

$$v'(y) = c_1\, \mathrm{e}^{-\int\left(\frac{1}{2y} + \frac{1}{4D}\right)dy} = c_1 y^{-\frac{1}{2}}\, \mathrm{e}^{-\frac{y}{4D}},$$

and integrating once more that

$$v(y) = c_1 \int_0^y z^{-\frac{1}{2}}\, \mathrm{e}^{-\frac{z}{4D}}\, dz + c_2.$$

Thus, the diffusion equation $u_t = Du_{xx}$ has general solution

$$u(x,t) = v\left(\frac{x^2}{t}\right) = c_1 \int_0^{\frac{x^2}{t}} z^{-\frac{1}{2}}\, \mathrm{e}^{-\frac{z}{4D}}\, dz + c_2,$$

with two integration constants c_1 and c_2.

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We now observe that if $u(x,t)$ is a solution of the diffusion equation, then so is $u_x(x,t)$ by linearity of the equation. For the general solution found above this yields another solution $U(x,t) = u_x(x,t)$:

$$U(x,t) = c_1 \frac{2x}{t} \left(\frac{x^2}{t}\right)^{-\frac{1}{2}} \mathrm{e}^{-\frac{x^2}{4Dt}} = c_1 \frac{2}{\sqrt{t}} \mathrm{e}^{-\frac{x^2}{4Dt}} .$$

The integration constant c_1 is chosen such that $U(x,t)$ satisfies:

$$\int_{-\infty}^{+\infty} U(x,t)dx = 1,$$

for all $t > 0$.

This constraint is motivated by the fact that it also holds for $t = 0$:

$$\int_{-\infty}^{+\infty} u_0(x)dx = \int_{-\infty}^{+\infty} \delta_0(x)dx = 1,$$

and that the diffusion equation models movement of individuals (not deaths or births), so that the total population should not change over time.

Thus, to find c_1, we let

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} \frac{2c_1}{\sqrt{t}} \mathrm{e}^{-\frac{x^2}{4Dt}} \, dx \\ &= \frac{4c_1}{\sqrt{t}} \int_0^{+\infty} \mathrm{e}^{-\left(\frac{x}{2\sqrt{Dt}}\right)^2} dx \\ &= 8\sqrt{D}c_1 \int_0^{+\infty} \mathrm{e}^{-z^2} \, dz \\ &= 8\sqrt{D}c_1 \frac{\sqrt{\pi}}{2} \end{aligned}$$

In the first step we used the fact that the integrand is an even function (so the integral equals twice the integral of the function over the interval $[0,+\infty)$). In the second step we used the substitution $z = x/(2\sqrt{Dt})$, and in the last step we used the famous integral[1]

$$\int_0^{+\infty} \mathrm{e}^{-z^2} \, dz = \frac{\sqrt{\pi}}{2}$$

Solving for c_1:

$$c_1 = \frac{1}{4\sqrt{D\pi}},$$

and plugging this back into the formula of $U(x,t)$, we finally arrive at the fundamental solution.

HW Now do problem # 4.5.2 of our text. Part (a) has of course just been shown in these notes, but you should not use this information when solving this part. Replace part (b) by the following: " (b) Make sure that $g(x,t) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$, and show that

$$\lim_{t \to 0+} g(x,t) = 0 \text{ if } x \neq 0, \text{ and } \lim_{t \to 0+} g(x,t) = +\infty \text{ if } x = 0.$$

Moreover, show that

$$\lim_{x \to \pm\infty} g(x,t) = 0, \text{ for all } t > 0."$$

HW This problem justifies the claim made above that the total population does not change over time. Let $u(x,t)$ be a solution of the diffusion equation $u_t = u_{xx}$ with the property that for all $t > 0$:

$$\lim_{x \to \pm\infty} u(x,t) = \lim_{x \to \pm\infty} u_x(x,t) = 0.$$

[1] Proof: The trick is to first calculate the *square* of this integral, and then to go to polar coordinates:

$$\left(\int_0^{\infty} \mathrm{e}^{-z^2} \, dz\right)^2 = \left(\int_0^{\infty} \mathrm{e}^{-x^2} \, dx\right)\left(\int_0^{\infty} \mathrm{e}^{-y^2} \, dy\right) = \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-x^2-y^2} \, dxdy = \int_0^{\frac{\pi}{2}}\int_0^{+\infty} \mathrm{e}^{-r^2} \, rdrd\theta = \frac{\pi}{4}.$$

[This is a very natural assumption. Notice in particular that these properties hold for the fundamental solution.]

Show that

$$\frac{d}{dt}\left(\int_{-\infty}^{+\infty} u(x,t)dx\right) = 0,$$

from which the claim follows.

Hint: Rewrite the above quantity as $\int_{-\infty}^{+\infty} u_t(x,t)dx$, and now use the diffusion equation, integration by parts and some of the conditions on $u(x,t)$ given above.