Notes on cobbwebbing

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Here we will consider the scalar nonlinear discrete-time system:

$$x_{n+1} = f(x_n), \;\; x_n \in [0, +\infty), \;\; n = 0, 1, \dots \tag{1}$$

under certain assumptions about the function f. We say that x is a fixed point of (1) if $f(x) = x$. Denote the kth composition of f with itself as f^k (e.g. $f^2 = f \circ f$), and thus $x_n = f^n(x_0)$. We call a point y a period 2 point for (1) if $f^2(y) = y \neq f(y)$.

f is non-decreasing

Let $f : [0, +\infty) \to [0, +\infty)$ be continuous and non-decreasing, i.e. $x < y$ implies that $f(x) \leq f(y)$.

We have the following result.

Theorem 1. *Every bounded solution sequence x_n converges to a fixed point as $n \to +\infty$.*

Proof. Let x_n be a bounded solution sequence. If $f(x_0) = x_0$, then x_0 is a fixed point. If $x_0 < f(x_0)$, then $x_n = f^n(x_0)$ is an non-decreasing sequence which is bounded, and therefore it converges to, say x^*. Let's show that x^* must be a fixed point. Since f is continuous, the sequence $f^{n+1}(x_0)$ converges to $f(x^*)$. But since $f^{n+1}(x_0)$ is a subsequence of $f^n(x_0)$, it follows that $f(x^*)$ and x^* must be equal. If $x_0 > f(x_0)$, then $f^n(x_0)$ is non-increasing and bounded below (by 0), and hence it converges to a fixed point as well. □

Remark 1. What is remarkable about this result is that the fate of all solution sequences is known: either they converge to some fixed point, or they grow unbounded. Compare this to the logistic map we discussed in class, where very complicated behavior could occur.

f is non-increasing

Let $f : [0, +\infty) \to [0, +\infty)$ be continuous and non-increasing, i.e. $x < y$ implies that $f(x) \geq f(y)$.

We have the following result.

Theorem 2. *There is a unique fixed point x^* for* (1)*, and every solution sequence x_n converges to x^* as $n \to +\infty$ if and only if* (1) *has no period* 2 *points.*

Proof. Assume that f is not the zero map, for otherwise the result would be obvious (all points are mapped to 0 in one step), and thus $f(0) > 0$. Since f is non-increasing it follows that $f(f(0)) \leq f(0)$. Setting $g(x) = f(x) - x$ we see that $g(0) > 0$ and $g(f(0)) \leq f(0)$, and thus by the mean value theorem there is some $x^* \in (0, f(0)]$ such that $g(x^*) = 0$. In other words x^* is a fixed point for (1). Moreover, x^* is the unique fixed point, for if $x_1^* < x_2^*$ where two fixed points, then $f(x_1^*) = x_1^* < x_2^* = f(x_2^*)$ would contradict that f is non-increasing.

If every solution sequence x_n of (1) converges to x^*, then clearly there cannot be any period 2 points. Let us prove the converse. Pick some $x_0 \in [0, +\infty)$. We need to show that x_n converges to x^* as $n \to +\infty$.

Case 1: $x_0 < f(x_0)$. Since f is non-increasing it follows that $f(x_0) \geq f^2(x_0)$. Since there are no period 2 points this implies that either $x_0 < f^2(x_0)$ or $x_0 > f^2(x_0)$. Assume first that $x_0 < f^2(x_0)$. Since f is non-increasing, it follows that f^2 is non-decreasing and therefore the sequence $f^{2k}(x_0)$ is non-decreasing as well. Since it is bounded above (by $f(0)$, the largest value of f on $[0, +\infty)$), this sequence converges to some value x_e:

$$f^{2k}(x_0) \to x_e \text{ as } k \to +\infty. \tag{2}$$

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Since f is continuous, it follows that the sequence $f^{2k+1}(x_0)$ converges to $f(x_e)$ as $k \to +\infty$. Similarly, the sequence $f^{2(k+1)}(x_0)$ converges to $f^2(x_e)$ which must be equal to x_e by (2). Finally, $f(x_e) = x_e$, for otherwise x_e would be a period 2 point. Thus, x_e is a fixed point, and since there is only one fixed point we conclude that $x^e = x^*$. Since both $f^{2k}(x_0)$ and $f^{2k+1}(x_0)$ converge to x^* as $k \to +\infty$, it is clear that $f^n(x_0)$ converges to x^* as well $n \to +\infty$. If $x_0 > f^2(x_0)$, the proof is similar although now $f^{2k}(x_0)$ is non-increasing (can you complete the steps?).

Case 2: $x_0 > f(x_0)$. The proof is similar as the proof of Case 1. (complete the steps)

Case 3: $x_0 = f(x_0)$. This implies that $x_0 = x^*$ since x^* is the only fixed point. The conclusion is obvious in this case. $\square$

Remark 2. It is not hard to find examples of non-increasing maps f with the property that (1) has period two points. For instance, let $f(x) = 1 - x$ when $x \in [0, 1]$ and $f(x) = 0$ when $x > 1$. Then every point in $[0, 1]$, except for $1/2$, is a period 2 point. Indeed, when $x \in [0, 1]$, then $f^2(x) = 1 - (1 - x) = x$.

Remark 3. What is remarkable about this result is that if you can rule out period 2 points, then you can conclude convergence of all solution sequences to the same fixed point. Again, compare this to the logistic map we discussed in class, where very complicated behavior could occur.