Exam 1: MAP 4484*

February 6, 2009

Name:
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This is a **closed book** exam and the use of calculators is **not** allowed.

1. A population model is described by the following equation:

$$x_{n+1} = \frac{2}{1+x_n}$$

- Calculate all fixed points and all period 2 points (if any).
  Fixed points are solutions of:
  $$x = \frac{2}{1+x},$$
  or equivalently of:
  $$x^2 + x - 2 = 0.$$
  This quadratic equation has two solutions:
  $$x_{1,2} = \frac{-1 \pm 3}{2} = 1, -2$$
  We discard the negative solution, and thus there is a single fixed point:
  $$x^* = 1.$$
  Period two points must satisfy:
  $$x = \frac{2}{1 + \frac{2}{1+x}},$$
  which reduces to
  $$x^2 + x - 2 = 0,$$
  the same quadratic equation we solved before. Thus, there are no period two points, as the only solution of the above quadratic equation only yields the fixed point determined earlier.
- What happens to the solution sequences as $n \to \infty$? (**Hint**: Use a global result from our cobwebbing notes)
  Since the function $f(x) = 2/(1+x)$ is decreasing and bounded, and since there are no period two points, we conclude that all solutions converge to the unique fixed point $x^* = 1$.
- Notice that if $x_0 = 0$, then $x_1 = 2$. Thus from no individuals, we go to 2 individuals. What could cause this?
  Immigration,...

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2. A population model is described by the following equation:

$$x_{n+1} = \frac{3x_n}{1+2x_n^2} x_n$$

- Determine all fixed points and determine their stability based on linearization.

  Fixed points: $x = 0$ clearly, and positive solutions to:

  $$1 = \frac{3x}{1+2x^2},$$

  or equivalently to the quadratic equation:

  $$2x^2 - 3x + 1 = 0.$$

  Solutions are:

  $$x = \frac{3 \pm 1}{4} = 1, \frac{1}{2}.$$

  Linearizing:

  $$f' = \frac{6x(1+2x^2) - 3x^2(4x)}{(1+2x^2)^2} = \frac{6x}{(1+2x^2)^2},$$

  yields:

  $$f'(1) = \frac{6}{9} \text{ so } x = 1 \text{ is asymptotically stable and}$$
  $$f'\left(\frac{1}{2}\right) = \frac{3}{\left(1+\frac{1}{2}\right)^2} = \frac{4}{3} \text{ so } x = \frac{1}{2} \text{ is unstable}$$
  $$f'(0) = 0, \text{ so } x = 0 \text{ is asymptotically stable.}$$

- Perform cobwebbing, and verify if your results agree with the stability analysis carried out in the previous item.

  The function $\frac{3x_n}{1+2x_n^2} x_n$ is increasing, zero at zero, and has limit $3/2$ as $x \to \infty$. The diagonal intersects this graph in the three fixed points, $0, \frac{1}{2}$ and 1.

  All solutions in $(0, \frac{1}{2})$ converge to 0. All solutions in $(\frac{1}{2}, 1)$ converge to 1, and all solutions in $(1, +\infty)$ converge to 1.

- What happens to the solution sequences as $n \to \infty$? (**Hint**: Use a global result from our cobwebbing notes)

  All solutions in $(0, \frac{1}{2})$ converge to 0. All solutions in $(\frac{1}{2}, 1)$ converge to 1, and all solutions in $(1, +\infty)$ converge to 1.

- Explain why this model could be called a "population switch".

  Depending on the value of the initial population, solutions either converge to 0 or to 1 (except if they start in the fixed point $x = \frac{1}{2}$ of course).

3. The following system models a population of parasites and hosts:

$$\begin{aligned} x_{n+1} &= 2\,\mathrm{e}^{-y_n}\,x_n \\ y_{n+1} &= (1-\mathrm{e}^{-y_n})x_n \end{aligned}$$

Note: A more general form of this system is described in our textbook.

- Which variable represents the number of hosts at time n, x_n or y_n? Explain your answer.
  x_n are hosts, as the presence of parasites diminishes growth from 2 to $2\,\mathrm{e}^{-y_n}$.
- Find all fixed points, and determine their stability based on linearization. (**Hint**: You may use the fact that $\ln(2) > \frac{1}{2}$)
  $(0,0)$ is clearly a fixed point. Other fixed points solve:

$$\begin{aligned} 1 &= 2\,\mathrm{e}^{-y} \Rightarrow y = \ln(2) \\ y &= (1-\mathrm{e}^{-y})x \Rightarrow x = \frac{\ln(2)}{\frac{1}{2}} = 2\ln(2) \end{aligned}$$

  so $(2\ln(2), \ln(2))$ is a second fixed point.
  Jacobian matrix:

$$Jac(x,y) = \begin{pmatrix} 2\,\mathrm{e}^{-y} & -2\,\mathrm{e}^{-y}\,x \\ 1-\mathrm{e}^{-y} & +\mathrm{e}^{-y}\,x \end{pmatrix}$$

  Evaluating at the fixed points:

$$Jac(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow (0,0) \text{ is unstable by eigenvalue } 2,$$

$$Jac(2\ln(2), \ln(2)) = \begin{pmatrix} 1 & -2\ln(2) \\ \frac{1}{2} & \ln(2) \end{pmatrix}$$

  Checking the first Jury condition:

  $|tr| = |1+\ln(2)| = 1+\ln(2) < 1+det = 1+(\ln(2)+\ln(2)) = 1+2\ln(2)$ is clearly satisfied.

  The second Jury condition:

$$1 + det = 1 + 2\ln(2) < 2 \text{ fails by the } \textbf{Hint}.$$

  Thus, $(2\ln(2), \ln(2))$ is unstable as well.