Homework assignment 1*

Due date: Wednesday February 8, 2008.

1. (# 1.2.9) Consider a thin one-dimensional rod whose lateral surface area is *not insulated*. (a) Assuming exact conservation of energy, and assuming that the amount of heat energy flowing out laterally at x per lateral unit area and per unit of time is $w(x,t)$, derive the PDE describing the temperature $u(x,t)$. (b) Assuming that $w(x,t)$ is proportional to the difference of the inside temperature $u(x,t)$ and the outside temperature $\gamma(x,t)$ with positive proportionality factor $h(x)$, show that the PDE for $u(x,t)$ becomes:

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}(x,t)\right) + Q(x,t) - \frac{P}{A}h(x)(u(x,t) - \gamma(x,t)),$$

   where P is the lateral perimeter. (d) Specialize the previous PDE to the case of a rod with constant thermal properties, without internal heat sources, constant zero outside temperature and constant circular cross section.

2. (# 4.2.1) This problem shows that in case the external force on a string is due to gravity alone, it sufficies to study the simpler wave equation $\partial^2 u/\partial t^2 = c^2 \partial^2 u/\partial x^2$: (a) Assuming a uniform string (i.e. $\rho_0(x)$ is constant), $Q(x,t) = -g$ and starting from:

$$\rho_0 \frac{\partial^2 u}{\partial t^2}(x,t) = T_0 \frac{\partial^2 u}{\partial x^2}(x,t) + Q(x,t)\rho_0,$$

   with

$$u(0) = u(L) = 0,$$

   determine the sagged equilibrium shape $u_E(x)$ of the string. Draw $u_E(x)$. (b) Show that the difference $u(x,t) - u_E(x)$ satisfies the wave equation.

3. (# 1.5.12) Let $u(x,y,z,t) = u(r,t)$ (where $r = \sqrt{x^2+y^2+z^2}$, so the temperature is spherically symmetric) and suppose that heat flows between two concentric spheres of radius a and b respectively. (a) Show that the total heat energy is

$$4\pi \int_a^b c(r)\rho(r)u(r,t)r^2 dr.$$

   (b) Show that the heat energy per time unit leaving through the shell at b is

$$-4\pi b^2 K_0(b)\frac{\partial u}{\partial r}(b,t),$$

   and derive a similar formula for the shell at a (but now *entering* the shell at a). (c) Using (a) and (b) and assuming that the thermal characteristics are uniform, derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t}(r,t) = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}(r,t)\right),$$

   where $k = K_0/(c\rho)$ is the thermal diffusivity.

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4. (# 2.3.4) Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x).$$

(a) What is the total heat energy in the rod as a function of time? (No need to solve the equation, since we did it in class) (b) What is the flow of heat energy per unit area and per unit of time leaving the rod at $x = 0$? At $x = L$? (c) What relationship should exist between (a) and (b)? Don't just use the words from the solution key. Instead, provide formulas explaining this relationship.

5. (# 2.3.8) Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u, \quad u(0,t) = u(L,t) = 0.$$

where $\alpha > 0$ is a constant. This corresponds for instance to a rod which is not insulated laterally (see problem 1 above with 0 degree outside temperature). (a) Find all possible equilibrium temperature distributions. Briefly explain your result physically. (b) When adding an initial condition $u(x,0) = f(x)$, solve the above time-dependent problem. Examine $\lim_{t\to\infty} u(x,t)$ and compare to part (a).

6. (# 2.5.5 (b)) Solve Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad \frac{\partial u}{\partial \theta}(r,0) = \frac{\partial u}{\partial \theta}(r,\frac{\pi}{2}) = 0, \quad u(1,\theta) = f(\theta).$$

in a quarter circle of radius 1 ($\theta \in [0,\pi/2]$ and $r \in [0,1]$).

7. (# 2.5.15(d)) Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the semi-infinite strip ($(x,y) \in (0,\infty) \times (0,H)$), with boundary conditions:

$$\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,H) = 0, \quad \frac{\partial u}{\partial x}(0,y) = f(y).$$

Show that the solution exists only if $\int_0^H f(y)dy = 0$.