Practice Exam 3: MAP 4305*

1. Given is a fundamental matrix solution $X(t)$ of a system $\dot{x} = Ax$ where A is a (unknown, at least for now) 2 by 2 matrix:
$$X(t) = \begin{pmatrix} e^{5t} & e^{5t} + e^{-t} \\ e^{5t} & e^{5t} - e^{-t} \end{pmatrix}.$$
What is e^{tA}, and what is A?

2. Find the matrix exponential of:
$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

3. Solve the following initial value problem (use variation of constants formula):
$$\dot{x} = Ax + b, \;\; x(0) = x_0$$
where
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ t \end{pmatrix}, x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

4. Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a twice continuously differentiable function. Consider the system
$$\begin{aligned} \dot{x} &= V_x(x, y) \\ \dot{y} &= V_y(x, y) \end{aligned}$$
Let (x^*, y^*) be a critical point of V (ie $V_x(x^*, y^*) = V_y(x^*, y^*) = 0$). Then (x^*, y^*) is clearly an equilibrium point of the system. Can it be a spiral (stable or unstable) or a center?

5. Show that there are no non-constant periodic solutions for:
$$\begin{aligned} \dot{x} &= 2x - y + x^3y^2 \\ \dot{y} &= x - y \end{aligned}$$

6. Show that the following system has one stable and one unstable (non-trivial) limit cycle. Where are these limit cycles?
$$\begin{aligned} \dot{x} &= x(r^2 - 3r + 2) - y \\ \dot{y} &= y(r^2 - 3r + 2) + x, \end{aligned}$$
where $r = \sqrt{x^2 + y^2}$.
(**Hint**: Use polar coordinates)

7. Complete the discussion in class regarding the van der Pol oscillator by verifying that the region containing the origin and bounded by the line segments $L_1 : y = x+6, \; x \in [-3, 0]$, $L_2 : y = 6, \; x \in [0, 3]$, $L_3 : x = 3, \; y \in [-3.6]$, $L_4 : y = x - 6, \; x \in [0, 3]$, $L_5 : y = -6, \; x \in [-3, 0]$, and $L_6 : x = -3, \; y \in [-6, 3]$ is a trapping region. Recall the equations for the Van der Pol oscillator:
$$\begin{aligned} \dot{x} &= y + x - x^3/3 \\ \dot{y} &= -x \end{aligned}$$

*Instructor: Patrick De Leenheer.