Homework assignment 4*

March 14, 2006

1. Carry out the phase plane analysis for the Lotka-Volterra system:

$$\begin{aligned} \dot{y}_1 &= r_1 y_1[1 - y_1 - \alpha_{12} y_2] \\ \dot{y}_2 &= r_2 y_2[1 - y_2 - \alpha_{21} y_1] \end{aligned}$$

   assuming that $0 < \alpha_{12}, \alpha_{21} < 1$. Draw the nullclines, steady states and indicate the direction of the vector field in the different parts of the state space. Based on linearization, discuss stability of all steady states. Show that all solutions in the first quadrant converge to some steady state and show that almost all solutions converge to a particular steady state in the interior of the first quadrant. For the latter solutions this implies that in the long run, the species will coexist.

2. In class we analyzed the Brusselator:

$$\begin{aligned} \dot{x} &= 1 - (\beta + 1)x + \alpha x^2 y \\ \dot{y} &= -\alpha x^2 y + \beta x \end{aligned}$$

   Using the software of your choice, perform simulations for this system assuming that we fix $a = 2$, and let b range from 2 to 4 with increments of 0.5. Consider the same three to four initial conditions for all cases. Plot the solutions in each case in a single phase plane diagram. Make sure to provide a print-out of the commands you used to generate the simulations and plots.

   [For your convenience, I have put a sample of a Mathematica notebook on the class webpage for a similar problem. To execute a command in Mathematica, you need to simultaneously press the shift and enter button.]

3. This problem considers the Van der Pol oscillator:

$$\begin{aligned} \dot{x} &= x + y - x^3/3 \\ \dot{y} &= -x \end{aligned}$$

   Here $(x, y) \in \mathbb{R}^2$. Show that $(0, 0)$ is the only steady state, and that it is a repellor (linearization has two eigenvalues with positive real part). Show that the polytope $\{(x, y) \in \mathbb{R}^2 \mid x \in [-3, 3],\ y \in [-6, 6], y \leq x + 6, y \geq x - 6\}$ is a trapping region. Conclude that the Van der Pol oscillator must have a periodic solution.

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