Homework assignment 2*

February 3, 2006

1. Explain the occurence of an inflection point in the solutions of the Verhulst equation that have initial conditions in the interval $(0, K/2)$, where K is the carrying capacity. Do this without using the explicit expression for the solution we derived in class.

2. The Allee effect says that for very small population numbers, the population will not be viable and go extinct. That is, there is some small $\epsilon > 0$ such that if $N(0) \in (0, \epsilon)$, then $N(t) \to 0$ as $t \to \infty$. Write down the simplest possible continuous-time population model $dN/dt = NF(N)$ you can think of that exhibits the Allee effect, but behaves like the Verhulst equation otherwise (that is, there is some carrying capacity K to which all solutions -except for those in $[0, \epsilon]$- converge as $t \to \infty$).

3. Verify all the calculations we skipped in class regarding the interpretation of the inverse of the dilution rate in the chemostat model.

4. In class we stated -without proof- the linearization theorem for a steady state N^* of a scalar differential equation $dN/dt = f(N)$. Recall that no conclusion was reached if $f'(N^*) = 0$.
   - State the definition of stability and asymptotic stability of a steady state of this equation.
   - Provide 2 examples that explain why the linearization theorem does not yield a conclusion in case $f'(N^*) = 0$.

5. We studied the linear delay equation:
$$\frac{dN}{dt}(t) = -N(t - \tau)$$
and focused on the roots of its associated characteristic equation:
$$\lambda + \mathrm{e}^{-\lambda\tau} = 0, \quad \lambda \in \mathbb{C}.$$
We showed that for *some* values of τ in $(\frac{\pi}{2}, \infty)$, there are roots in the open right half plane. Choose one from:
   - (Challenge!) Prove that there are roots in the open right half plane for *every* value of τ in $(\frac{\pi}{2}, \infty)$.
   - Using a computer and your preferred software, plot the roots in the complex plane for all $\tau \in (0, \infty)$. (Please include a print-out of the code you used to generated your plot).

6. This problem concerns Leslie's age-structured model $x(t + 1) = Ax(t)$.
   - When is A irreducible? When is it primitive? (give necessary and sufficient conditions for both cases)?
   - Show that A is primitive if $f_1, f_n > 0$ and $p_i > 0$ for all i. Show that A is primitive if $f_{n-1}, f_n > 0$ and $p_i > 0$ for all i.
   - Write the general solution of Leslie's model assuming that $f_i = 0$ for $i = 1, ..., n - 1$ but $f_n = 1$ and $p_j = 1$ for all j. (Notice that A is a permutation matrix in this case, that everyone always survives to the next age class, and that only the oldest group gives birth). What happens to the fractions of the age classes as $t \to \infty$? Is there convergence? This should be immediately obvious if you think about the effect of multiplying a permutation matrix and a vector. Do you see it? Explain.

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