Exercise 9.17(6) (Contraction mapping)

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Proposition 1. *Let $f : I \to I$ be continuous on the closed interval I and assume that there exists $0 < \alpha < 1$ such that:*

$$\forall x,\, y \in I : \; |f(x) - f(y)| \leq \alpha |x - y| \tag{1}$$

Then f is continuous.

Pick $x_1 \in I$ and construct the sequence $\{x_n\}$ as follows:

$$x_n = f(x_{n-1}),\ \forall n > 1.$$

Then there is some $x^* \in I$ such that $x_n \to x^*$ as $n \to \infty$. Moreover, $f(x^*) = x^*$ (that is, f has a fixed point in I).

Proof. Fix $x, y \in I$ and $\epsilon > 0$, and choose $\delta = \epsilon/\alpha$. Then if $|x - y| < \delta$, it follows from (1) that $|f(x) - f(y)| \leq \alpha|x - y|$ and thus that $|f(x) - f(y)| < \alpha\delta = \epsilon$. This implies that f is continuous on I.

Pick $x_1 \in I$ and construct the sequence $\{x_n\}$ as outlined above. We will show that $\{x_n\}$ is a Cauchy sequence. If we do that, then it follows that $x_n \to x^*$ as $n \to \infty$ for some x^*. Since I is closed, there must hold that $x^* \in I$. Now, since f is continuous on I, it follows that $f(x_n) \to f(x^*)$ as $n \to \infty$. But $f(x_n) = x_{n-1}$ and $x_{n-1} \to x^*$ as $n \to \infty$. This implies that $f(x^*) = x^*$.

So let us conclude by showing that $\{x_n\}$ is a Cauchy sequence. To do so, we must prove that

$$\forall \epsilon > 0,\ \exists N, \text{ such that if } n, m > N, \text{ then } |f(x_n) - f(x_m)| < \epsilon.$$

Fix $\epsilon > 0$ and assume (without loss of generality) that $n > m$. Then by repeated application of the triangle inequality,

$$|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|. \tag{2}$$

An application of (1) shows that if $n > 2$, then

$$|x_n - x_{n-1}| = |f(x_{n-1}) - f(x_{n-2})| \leq \alpha |x_{n-1} - x_{n-2}|,$$

and thus that

$$|x_n - x_{n-1}| \leq \alpha^{n-2} |x_2 - x_1|. \tag{3}$$

Therefore, applying (3) to each term in (2), shows that

$$|x_n - x_m| \leq (\alpha^{n-2} + \alpha^{n-3} + \cdots + \alpha^{(m+1)-2})|x_2 - x_1| = \alpha^{m-1}(\alpha^{(n-2)-(m-1)} + \alpha^{(n-3)-(m-1)} + \cdots + 1)|x_2 - x_1|.$$

Since $\alpha \in (0, 1)$, the geometric series $\sum_{k=0}^{\infty} \alpha^k$ converges to $1/(1 - \alpha)$ and thus we have that

$$|x_n - x_m| \leq \alpha^{m-1} \frac{1}{1 - \alpha} |x_2 - x_1|.$$

We can choose N large enough so that

$$\alpha^{N-1} \frac{1}{1 - \alpha} |x_2 - x_1| < \epsilon.$$

Then is follows that for $n, m > N$,

$$|x_n - x_m| < \epsilon,$$

and thus that $\{x_n\}$ is a Cauchy sequence, concluding the proof of this theorem.

□