

Solutions HW 9

1. X_i = winnings on i^{th} bet

$$E(X_i) = (+1)\frac{18}{38} + (-1)\frac{20}{38} = -\frac{2}{38}$$

$$X = \sum_{i=1}^{19} X_i, \text{ so } E(X) = 19E(X_i) = -1$$

5. (Compare this to putting 10 balls into 6 boxes)

$X_i = \begin{cases} 1 & \text{if no stop at floor } i \text{ (} i=1, \dots, 6 \text{)} \\ 0 & \text{otherwise} \end{cases}$ Then $E(X_i) = (1 - \frac{1}{6})^{10} = (\frac{5}{6})^{10}$

$$X = \sum_{i=1}^6 X_i = \# \text{ of skipped-over floors, } E(X) = 6E(X_i) = 6(\frac{5}{6})^{10} \approx 5.03$$

10. X = outcome for die 1, Y = outcome for die 2 (both in $\{1, 2, \dots, 6\}$)

$Z = XY$, and since X and Y are independent (check!): $E(Z) = E(XY) = E(X)E(Y)$

$$\begin{aligned} \text{Var}(Z) &= E((Z - E(Z))^2) = E(Z^2) - (E(Z))^2 = \left(\frac{91}{6}\right)^2 - \left(\frac{21}{6}\right)^4 = \left(\sum_{k=1}^6 k \frac{1}{6}\right)^2 = \left(\frac{21}{6}\right)^2 \\ E(Z^2) &= E(X^2 Y^2) = \sum_{x,y=1}^6 x^2 y^2 P(X=x, Y=y) = \sum_{x=1}^6 x^2 P(X=x) \sum_{y=1}^6 y^2 P(Y=y) \\ &= E(X^2)E(Y^2) = \left(\sum_{k=1}^6 k^2 \frac{1}{6}\right)^2 = \left(\frac{91}{6}\right)^2 \end{aligned}$$

13. Z = # of students accepted = $\sum_{i=1}^{12} X_i + \sum_{i=1}^4 Y_i$, $X_i = \begin{cases} 1 & \text{if boy } i \text{ accepted} \\ 0 & \text{otherwise} \end{cases}$, $Y_i = \begin{cases} 1 & \text{if girl } i \text{ accepted} \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} E(X_i) &= P(X_i = 1) = 0.1 \\ E(Y_i) &= P(Y_i = 1) = 0.2 \\ \Rightarrow E(Z) &= 12E(X_i) + 4E(Y_i) = 1.2 + 0.8 = 2 \\ \text{Var}(Z) &= 12\text{Var}(X_i) + 4\text{Var}(Y_i) = 12(0.09) + 4(0.16) = 1.72 \end{aligned}$$

$\text{Var}(X_i) = E(X_i^2) - (E(X_i))^2 = 1^2(0.1) - (0.1)^2 = 0.09$
 $\text{Var}(Y_i) = \dots = 0.16$

$$(b) P(X = \sum_{i=1}^{12} X_i = 2, Y = \sum_{i=1}^4 Y_i = 0) = P(X=2)P(Y=0) = \binom{12}{2}(0.1)^2(0.9)^{10} \binom{4}{0}(0.2)^0(0.8)^4 = 0.0943$$

independence \swarrow of X_i, Y_j $X \sim \text{Bin}(12, 0.1), Y \sim \text{Bin}(4, 0.2)$

$$P(X=1, Y=1) = P(X=1)P(Y=1) = \binom{12}{1}(0.1)^1(0.9)^{11} + \binom{4}{1}(0.2)^1(0.8)^3 = 0.1542$$

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16. $X \in \{-2, -1, 0, 1, 2\}$, $P(X=i) = \frac{1}{5}$ so $E(X) = \frac{1}{5}(-2-1+0+1+2) = 0$, $\text{Var}(X) = 2$

$Y = X^2 \in \{0, 1, 4\}$, $P(Y=1) = P(Y=4) = \frac{2}{5}$, $P(Y=0) = \frac{1}{5}$ so $E(Y) = \frac{2}{5}(1+4) = 2$, $\text{Var}(Y) = \frac{14}{5}$

(a) $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0 - 0 = 0$ (since $E(XY) = E(X^3) = 0$)

(b) $\begin{array}{c|ccccc} X \backslash Y & -2 & -1 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1/5 & 0 & 0 \\ 1 & 0 & 1/5 & 0 & 1/5 & 0 \\ 4 & 1/5 & 0 & 0 & 0 & 1/5 \end{array}$ Ex: $P(X=0, Y=0) = P(X=0, X^2=0) = P(X=0) = 1/5$, $P(X=-1, Y=1) = P(X=-1, X^2=1) = P(X=-1) = 1/5$

From table: $0 = P(X=0, Y=1) \neq P(X=0)P(Y=1) = \frac{1}{5} \cdot \frac{2}{5}$, so X, Y not independent