HW8 Solutions

5.6 # 22, 25, 29, 30, 31

22| Weibull. $X \sim Exp(\lambda)$ so $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$

$Y \sim X^{1/\alpha}$, so $r(x) = x^{1/\alpha}$ strictly $\uparrow$ (assume $\alpha > 0$)

$s(y) = y^{\alpha}$

$\to Y$ has density $\lambda e^{-\lambda y^{\alpha}} \cdot (\alpha y^{\alpha-1}) = \lambda \alpha y^{\alpha-1} e^{-\lambda y^{\alpha}}$

25| X has density $f_X(x)$ for $x \in [-1, 1]$.

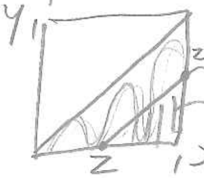
(a) Density for $Y = |X|$.

$$F_Y(y) = P(Y \leq y) = P(|X| \leq y) = \begin{cases} 0, & \text{if } y \leq 0 \\ \int_{-y}^{y} f_X(x)\,dx, & \text{if } 0 \leq y \leq 1 \\ \int_{-1}^{1} f_X(x)\,dx = 1, & \text{if } y > 1 \end{cases} \overset{\frac{d}{dy}}{\Longrightarrow} f_Y(y) = \begin{cases} 0, & \text{if } y \leq 0 \text{ or } y > 1 \\ f_X(y) + f_X(-y), & \text{if } 0 < y \leq 1 \end{cases}$$

(b) Density for $Z = X^2$.

$$F_Z(z) = P(Z \leq z) = P(X^2 \leq z) = \begin{cases} 0, & \text{if } z \leq 0 \\ \int_{-\sqrt{z}}^{+\sqrt{z}} f_X(x)\,dx, & \text{if } 0 < z \leq 1 \\ 1, & \text{if } z > 1 \end{cases} \overset{\frac{d}{dz}}{\Longrightarrow} f_Z(z) = \begin{cases} 0, & \text{if } z \leq 0 \text{ or } z > 1 \\ \frac{1}{2} z^{-\frac{1}{2}} \left(f(\sqrt{z}) + f(-\sqrt{z}) \right) & \text{if } 0 < z \leq 1 \end{cases}$$

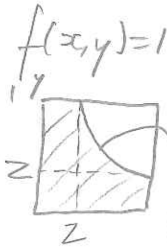
29| $f(x, y) = 2$ if $0 < y < x < 1$.



$$P(X - Y > z) = \begin{cases} \int_z^1 \int_0^{x-z} 2\,dy\,dx & \text{if } 0 \leq z \leq 1 \\ 1 & \text{for } z < 0 \\ 0 & \text{for } z > 1 \end{cases}$$

$$\overset{\text{calculus}}{\Longrightarrow} P(X - Y > z) = \begin{cases} (z-1)^2 & \text{if } 0 \leq z \leq 1 \\ 0 & \text{if } z < 0 \\ 1 & \text{if } z > 1 \end{cases}$$

30| $f(x, y) = 1$ on $[0,1] \times [0,1]$



$$P(XY \leq z) = \begin{cases} 0 & \text{if } z \leq 0 \\ 1 & \text{if } z \geq 1 \\ z \cdot 1 + (1-z)z + \int_z^1 \int_z^{\frac{z}{x}} 1\,dy\,dx & \text{if } 0 < z < 1 \end{cases}$$

$$\overset{\text{Calculus}}{\Longrightarrow} P(XY \leq z) = \begin{cases} 0 & \text{if } z \leq 0 \\ 1 & \text{if } z \geq 1 \\ 1 - z\ln z & \text{if } 0 < z < 1 \end{cases}$$

31| $P(|X - Y| \leq \frac{1}{4})$ = area of shaded region

$= 1 - 2 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4}$

$= \frac{7}{16}$