HW7 Solutions

3.5 : 62
5.6 : 3, 8, 10(a), 17

3.5 #62: (a) $P(X=1|Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{0.1}{0.1+0.2+0.3} = \frac{1}{6}$

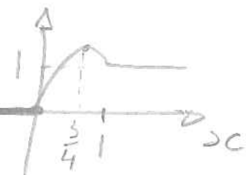
(b) $P(X=2|Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{0.15}{0.15+0.15+0} = \frac{15}{30} = \frac{1}{2}$

5.6 #3: $f(x) = \begin{cases} 6x(1-x), & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$ $\quad E(X) = \int_0^1 6x^2(1-x)\,dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_{x=0}^{x=1} = 6\left(\frac{1}{3}-\frac{1}{4}\right) = 6\frac{1}{12} = \frac{1}{2}$

$E(X^2) = \int_0^1 6x^3(1-x)\,dx = 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_{x=0}^{x=1} = 6\cdot\frac{1}{20} = \frac{3}{10}$

$Var(X) = E(X^2) - (E(X))^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{6-5}{20} = \frac{1}{20}$

5.6 #8: Plot of F :



$F(x) = \begin{cases} 3x - 2x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge 1 \end{cases}$

Note: $\frac{d}{dx}(3x - 2x^2) = 3 - 4x = 0$ if $x = \frac{3}{4}$

No, F is decreasing on $[\frac{3}{4}, 1]$

5.6 #10: $f(x) = \begin{cases} 4x^3, & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$ $\quad$ (a) $F(x) = \int_{-\infty}^{x} f(y)\,dy = \begin{cases} 0, & \text{if } x<0 \\ x^4, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \le 1 \end{cases}$

Note: F is continuous on $\mathbb{R}$

(b) $P(X < \frac{1}{2}) = P(X \le \frac{1}{2})$ since F is continuous at $x = \frac{1}{2}$, hence $P(X < \frac{1}{2}) = F(\frac{1}{2}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

(c) $P(\frac{1}{9} < X < \frac{1}{4}) = P(\frac{1}{9} < X \le \frac{1}{4})$ Since F is continuous at $x = \frac{1}{4}$, hence $P(\frac{1}{4} < X \le \frac{1}{4}) = F(\frac{1}{4}) - F(\frac{1}{9})$
$= \left(\frac{1}{4}\right)^4 - \left(\frac{1}{9}\right)^4$

5.6 #17: First, do 5.6 #16 (b): If $X_1 \dots X_n$ are independent RV, and $Z = \min\{X_1 \dots X_n\}$,
Then $P(Z \le z) = P(\min\{X_1 \dots X_n\} \le z) = 1 - P(\min\{X_1 \dots X_n\} > z) = 1 - P(X_1 > z, X_2 > z, \dots, X_n > z)$
$\underset{\text{independence}}{=} 1 - P(X_1 > z)P(X_2 > z) \dots P(X_n > z) = 1 - (1 - P(X_1 \le z))(1 - P(X_2 \le z)) \dots (1 - P(X_n \le z))$
$= 1 - (1 - F(z))^n$, where F is the distribution function of each X_i.

Now : X_i are independent and $Exp(\lambda)$, so $F_{X_i}(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \ge 0 \\ 0, & \text{otherwise} \end{cases}$

Let $Z = \min\{X_1 \dots X_n\}$. Then by 5.6 #16 b, Z has following distribution function:

$P(Z \le z) = \begin{cases} 1 - (1 - (1 - e^{-\lambda x}))^n = 1 - e^{-n\lambda x}, & \text{if } x \ge 0 \\ 0, & \text{otherwise} \end{cases}$

Thus Z is $Exp(n\lambda)$