Final Exam MTH361

1.

$X \backslash Y$	$Y=-1$	0	1
1	0	0.25	0
0	0.25	0.25	0.25

$\to E(X) = 0.25,\ E(Y) = -1(0.25) + 1(0.25) = 0$

- $Cov(X,Y) = E((X-E(X))(Y-E(Y))) = E(XY) - E(X)E(Y) = 0 - (0.25)0 = 0$
- $P(X=1, Y=1) = 0 \neq P(X=1)P(Y=1) = (0.25)(0.25) \to X, Y$ are not independent

3rd

2. Pick 2 cards from 52. $P(\text{card 2 is King}) = \frac{1}{13}$

2nd

3. $X \sim Pois(4)$. Chebyshev for $P(|X-4| \geq 2) \leq \frac{Var(X)}{2^2} = \frac{4}{4} = 1$

$\mu = \sigma^2$

Exact: $P(|X-4| \geq 2) = 1 - P(X=3) - P(X=4) - P(X=5)$

$= 1 - \sum_{k=3}^{5} P(X=k)$, $P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, k=0,1,\dots$

$\lambda = 4$

1st

4. $k \in \mathbb{Z}_+$ given. $T_k = \#$ of trials to get k successes. ($P(\text{success of 1 trial}) = p$)

$$T_k = \underbrace{(T_k - T_{k-1})}_{\text{geom. RV with } p \,\to\, \mu = \frac{1}{p}} + (T_{k-1} - T_{k-2}) + \dots + (T_1 - T_0) + T_0, \quad T_0 = 0$$

$E(T_k) = k \cdot \frac{1}{p}$

3rd

5. $P(N) = 0.6,\ P(S) = 0.5,\ P(P) = 0.4$

$P(N \cap S) = 0.3,\ P(S \cap P) = 0.2,\ P(N \cap P) = 0.1$

$P(N \cap S \cap P) = 0$

$P(N \cup S \cup P) = 0.6 + 0.5 + 0.4 - (0.3 + 0.2 + 0.1) + 0$
$= 0.9$

4th

6. $A =$ "student attends".

$P(A|M) = P(A|W) = 0.8$

$P(A|F) = 0.4$

$P(M) = P(W) = P(F) = \frac{1}{3}$

$$P(F|A) = \frac{P(A|F)P(F)}{P(A|F)P(F) + P(A|M)P(M) + P(A|W)P(W)}$$

$$= \frac{(0.4)\cancel{(1/3)}}{(0.4)\cancel{(1/3)} + (0.8)\cancel{(1/3)} + (0.8)\cancel{(1/3)}} = \frac{4}{20} = \frac{1}{5}$$

7. $X_1, \dots, X_n$ i.i.d. with distribution $F(x)$.

(a) $F_Y(z) = P(Y \leq z) = P(\min\{X_1, \dots, X_n\} \leq z) = 1 - P(\min\{X_1, \dots, X_n\} > z) = 1 - P(X_1 > z, X_2 > z, \dots, X_n > z)$

$\overset{\text{independence}}{=} 1 - P(X_1 > z)P(X_2 > z)\dots P(X_n > z) = 1 - (1 - P(X_1 \leq z))(1 - P(X_2 \leq z))\dots(1 - P(X_n \leq z)) = 1 - (1 - F(z))^n$

(b) $X_1, \dots, X_n$ i.i.d. and $Exp(\lambda) \to F(x) = 1 - e^{-\lambda x}, x \geq 0$ (0 otherwise)

$Y = \min(X_1, \dots, X_n) \Rightarrow$ By (a) $F_Y(z) = 1 - (1 - (1 - e^{-\lambda z}))^n = 1 - e^{-n\lambda z}, z \geq 0$ (0 otherwise). $Y \sim Exp(n\lambda)$

8. $P(W) = 0.45$, $P(C) = 0.55$

$$P(C \text{ win series}) = (0.55)^4 + 4(0.55)^4(0.45) + C_{5,3}(0.55)^4(0.45)^2 + C_{6,3}(0.55)^4(0.45)^3$$

$\underbrace{X\,X\,X\,X\,X\,X}_{3C's}\,\overset{C}{X}$ 7

$\underbrace{X\,X\,X\,X\,X}_{3C's}\,\overset{C}{X}$ 6

$\underbrace{X\,X\,X\,X}_{3C's}\,\overset{C}{X}$ 5

$\overset{C}{X}\,\overset{C}{X}\,\overset{C}{X}\,\overset{C}{X}$ 4

9. • Exact: S_n = # of games won $\sim Bin(n,p)$ with $n = 82$, $p = 0.9$

$$P(S_n \geq 73) = \sum_{k=73}^{82} C_{n,k}\, p^k (1-p)^{n-k} = \sum_{k=73}^{82} C_{82,k} (0.9)^k (0.1)^{82-k}$$

• $S_n = \sum_{i=1}^{82} X_i$, X_i is Bernoulli with $\mu = E(X_i) = 0.9$

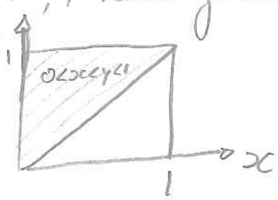
$\sigma^2 = Var(X_i) = (0.9)(1-0.9) = (0.9)(0.1) = 0.09$

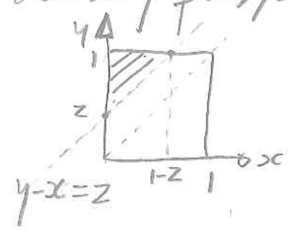
$\Rightarrow \sigma = (0.09)^{1/2} = \frac{3}{10} = 0.3$

$$P(S_n \geq 73) = P\Big(\underbrace{\frac{S_n - n\mu}{\sigma\sqrt{n}}}_{X} \geq \underbrace{\frac{73 - 82\cdot(0.9)}{(0.3)\sqrt{82}}}_{z}\Big) \overset{CLT}{\simeq} P(X \geq z) = 1 - \Phi(z)$$

where $z = \dfrac{73 - 82(0.9)}{(0.3)\sqrt{82}}$

10. X, Y have joint density $f(x,y) = 2$ on $0 < x < y < 1$





$$P(Y - X > z) = \begin{cases} \int_0^{1-z}\int_{x+z}^{1} 2\,dy\,dx & \text{if } 0 < z < 1. \\ 1 & \text{if } z \leq 0 \\ 0 & \text{if } z \geq 1 \end{cases}$$

$$\int_0^{1-z}\int_{x+z}^{1} 2\,dy\,dx = 2\int_0^{1-z} 1 - z - x\,dx = 2(1-z)^2 - \cancel{2}\frac{1}{\cancel{2}}(1-z)^2 = (1-z)^2$$

$$\Rightarrow P(Y - X > z) = \begin{cases} (1-z)^2 & \text{if } 0 < z < 1 \\ 1 & \text{if } z \leq 0 \\ 0 & \text{if } z \geq 1 \end{cases}$$