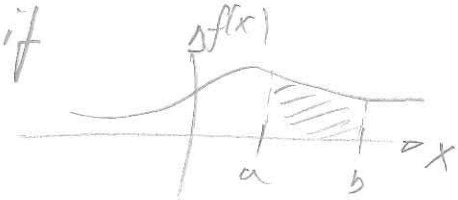
Lecture 18 11/2/16

Density functions for continuous distributions

↳ e.g. height of a person is a continuous RV.

Def [A continuous RV X has <u>density function</u> f if

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$



We recognize this as the area under the graph of f between $x=a$ and $x=$

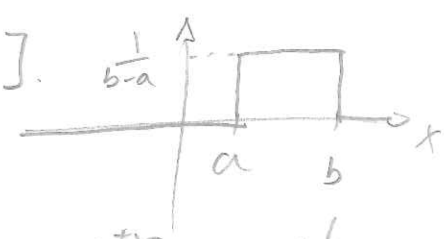
Remark 1: Informally, $f(x) \approx$ probability that $X=x$, although $P(X=x)$ is 0!

$$P(x \le X \le x+\Delta x) = \int_x^{x+\Delta x} f(y)\,dy \approx f(x)\,\Delta x$$

or $$f(x) = \frac{P(x \le X \le x+\Delta x)}{\Delta x}$$

Remark 2:
- $f(x)$ must be ≥ 0 for all x
- $\int_{-\infty}^{+\infty} f(x)\,dx$ must be 1

Ex 1: $\boxed{\text{Uniform distribution}}$ on $[a,b]$.



$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \text{ in } [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Indeed, $f(x) \geq 0$ for all x, and $\int_{-\infty}^{+\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = 1$

Ex 2: $\boxed{\text{Exponential distribution}}$ on $x \geq 0$

↳ $Exp(\lambda)$, where $\lambda > 0$ is a parameter

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Indeed, $f(x) \geq 0$ for all x and $\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_{x=0}^{x\to\infty} = 1$

Ex3 : Power Law on $x \geq 1$

↳ parameter $\rho > 1$

$$f(x) = \begin{cases} (\rho-1)x^{-\rho}, & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

5.6 #2

Expected Value, Variance

Recall for discrete RV :

$$\begin{cases} E(X) = \sum_x x P(X=x) \\ E(r(X)) = \sum_x r(x) P(X=x) \\ Var(X) = E((X-E(X))^2) = E(X^2) - (E(X))^2 \end{cases}$$

For continuous RV :

$$E(X) = \int x f(x) dx$$

$$E(r(X)) = \int r(x) f(x) dx$$

$$Var(X) = E((X-E(X))^2) = \int (x - E(X))^2 f(x) dx$$

$$= \int x^2 f(x) dx - (E(X))^2 \underbrace{\int f(x) dx}_{1}$$

$$= E(X^2) - (E(X))^2, \text{ same !}$$

Ex : - Uniform

- Exponential ⤳ integration by parts

- Power Law $E(X) = (\rho-1) \int_1^\infty x^{-\rho+1} dx = \begin{cases} (\rho-1) \frac{x^{-\rho+2}}{-\rho+2} \Big|_1^\infty = \frac{\rho-1}{\rho-2} & \text{if } \rho > 2 \\ (\rho-1) \ln x \Big|_1^\infty = \infty & \text{if } \rho = 2 \\ (\rho-1) \frac{x^{-\rho+2}}{-\rho+2} \Big|_1^\infty = \infty & \text{if } 1 < \rho < 2 \end{cases}$

Similarly, $E(X^2) = \begin{cases} \frac{\rho-1}{\rho-3} & \text{if } \rho > 3 \\ \infty & \text{if } 1 < \rho \leq 3 \end{cases}$

Do 5.6 #5

→ [illegible] tail distribution

Lecture 19/20 11/4/16

Def: Distribution function of a RV X:

$$F(x) = P(-\infty < X \leq x) = \int_{-\infty}^{x} f(y)\,dy$$, ie integral of density funct. from $-\infty$ to x.

Note: $\begin{cases} \lim_{x \to -\infty} F(x) = 0 \\ \lim_{x \to +\infty} F(x) = 1 \end{cases}$

Remark: $$P(a < X \leq b) = \int_a^b f(y)\,dy = \int_{-\infty}^{b} f(y)\,dy - \int_{-\infty}^{a} f(y)\,dy = F(b) - F(a)$$

Ex1 Uniform on $[a,b]$ $f(x) = \frac{1}{b-a}$ on $[a,b]$ $\to$ $F(x) = \begin{cases} \frac{1}{b-a}(x-a) & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$

Ex2: $Exp(\lambda)$ $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$ $\to$ $F(x) = \begin{cases} \int_0^x \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Memoryless property: Suppose $X \sim Exp(\lambda)$

Fix $t > 0$. Then $P(X > t) = 1 - P(X \leq t) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$!

Now let $s \geq 0$. Then $P(X > t+s \mid X > t) = \dfrac{P(X > t+s)}{P(X > t)} = \dfrac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s}$
$= P(X > s)$

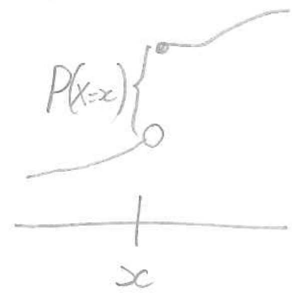
"Given that you've been waiting t units of time, the probability that you must wait s units of time, will be the same as if you had not waited at all"

Properties of Distribution functions $F(x)=\int_{-\infty} f(y)dy$

(i) $x_1 < x_2 \Rightarrow F(x_1) \le F(x_2)$ (nondecreasing)

(ii)-(iii) (see Remark) limits

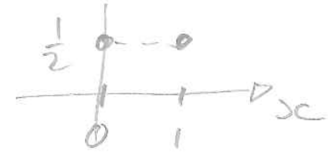
(iv) $\lim_{y \to x+} F(y) = F(x)$ (continuous from right)



(v) $\lim_{y \to x-} F(y) = P(Y<x)$

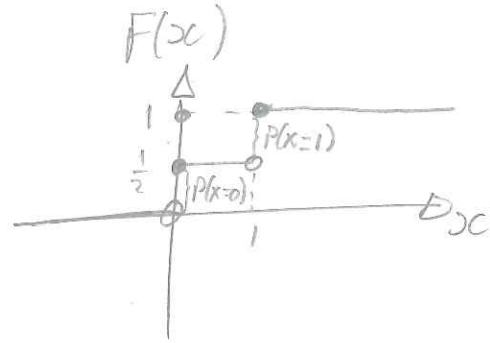
(vi) $\lim_{y \to x+} F(y) - \lim_{y \to x-} F(y) = P(X=x)$

Ex: Let $X = \begin{cases} 1 & \text{if H in 1 coin toss} \\ 0 & \text{otherwise} \end{cases}$



$P(X=0)=P(X=1)=\frac{1}{2}$

$$F(x)=\begin{cases} 0 & \text{if } x<0 \\ \frac{1}{2} & \text{if } 0\le x<1 \\ 1 & \text{if } 1\le x \end{cases}$$



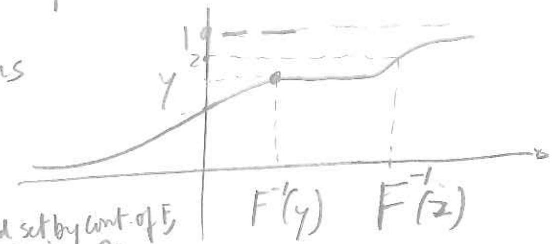
⊳ Calculated using: $F(x)=\sum_{k\le x} P(X=k)$

Lecture 20 11/7/16

Creating a uniform distribution, given an arbitrary distribution

<u>Thm</u>: Let X be such that $F(x)$ is continuous. Then $Y = F(X)$ is Uniform on $(0,1)$.

Pf: $F(x)$ is continuous



Define $F^{-1}(x) = \min \underbrace{\{r \mid F(r) \geq x\}}_{\text{a closed set by cont. of } F\text{, min may be } -\infty}$. Note: F^{-1} may be discontinuous, but it is increasing

Set $Y = F(X)$. We need the distribution f^{ion} of Y:

$$P(Y \leq y) = P(F(X) < y) \underset{F^{-1}\text{ is increasing}}{=} P(X < F^{-1}(y)) = F(F^{-1}(y)) \underset{F\text{ is cont.}}{=} y$$

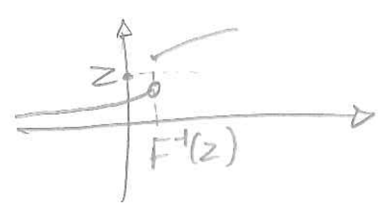
Thus $P(Y \leq y) = y$, ie Y has distribution of a Uniform RV on $(0,1)$.

<u>Converse</u>: **Creating an arbitrary distribution, starting from a Uniform distribution**

<u>Thm</u>: Let U be Uniform on $(0,1)$ and set $Y = F^{-1}(U)$, where F is an arbitrary distribution function (not necessarily continuous). Then Y has distribution function F.

Recall: $F^{-1}(x) = \min\{r \mid F(r) \geq x\}$

Thus for $0 < x < 1$: $F^{-1}(x) \leq y \Leftrightarrow x \leq F(y)$



$$\Rightarrow P(Y \leq y) = P(F^{-1}(U) \leq y) = P(U \leq F(y)) \underset{U\text{ is Uniform on }(0,1)}{=} F(y)$$

ie $F(y) = P(Y \leq y)$

<u>Ex</u>: Want to create an $\exp(\lambda)$ RV from Uniform.

$\hookrightarrow F(x) = 1 - e^{-\lambda x}$

Need F^{-1}: $1 - e^{-\lambda x} = y \Rightarrow x = \frac{-\ln(1-y)}{\lambda}$, hence $-\frac{\ln(1-U)}{\lambda}$ is $Exp(\lambda)$ when U is uniform on $(0,1)$

λ

Do 5.6 # 7, 16 → needed for HW # 17.

Skip: <u>Medians</u>

<u>Functions of RV</u>

<u>Ex</u>: Let $X \sim Exp(\lambda)$, set $Y = X^n$ for $n \geq 2$ positive integer.

$$P(Y \leq y) = P(X^n \leq y) = P(X \leq y^{\frac{1}{n}}) = 1 - e^{-\lambda y^{\frac{1}{n}}} \text{ if } y \geq 0.$$

→ density function of $Y = \frac{d}{dy}(1 - e^{-\lambda y^{\frac{1}{n}}}) = \begin{cases} \lambda \frac{1}{n} y^{\frac{1}{n}-1} e^{-\lambda y^{\frac{1}{n}}} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

<u>Thm</u>: Let X have density function $f(x)$.
Suppose a, b (possibly $-\infty, +\infty$ respectively) are such that $P(a < X < b) = 1$
Let $Y = r(X)$ for some given function $r : (a,b) \to (\alpha, \beta)$,
where r is <u>continuous</u> and <u>strictly increasing</u>.
Let $s : (\alpha, \beta) \to (a,b)$ be inverse function of r.
Then Y has density function $g(y) = \begin{cases} f(s(y))\, s'(y) & \text{for } \alpha < y < \beta \\ 0, & \text{otherwise} \end{cases}$

<u>Remark</u>: $g(y) = \begin{cases} -f(s(y))\, s'(y) & \text{for } \alpha < y < \beta \\ 0, & \text{otherwise} \end{cases}$ if r is <u>decreasing</u>

<u>Ex revisited</u>: $r(x) = x^n$ for $\underset{a}{0} < x < \underset{b}{+\infty}$. <u>Note</u>: $P(0 < X < \infty) = 1$ for $X \sim Exp(\lambda)$

Then $s(y) = y^{\frac{1}{n}}$ for $\underset{\alpha}{0} < y < \underset{\beta}{+\infty}$ (s is inverse function of r)

Let $Y = X^n$

By Thm, Y has density
$$g(y) = f(s(y))\, s'(y)$$
$$= \lambda e^{-\lambda s(y)} s'(y)$$
$$= \begin{cases} \lambda e^{-\lambda y^{\frac{1}{n}}} \cdot \frac{1}{n} y^{\frac{1}{n}-1} & \text{if } 0 < y < +\infty \\ 0, & \text{otherwise} \end{cases}$$

Do 5.6 # 18, 21

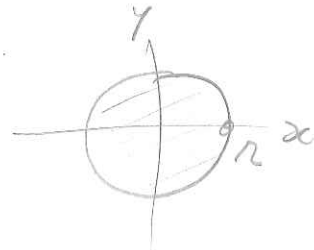
Lecture 21 11/9/16

Joint Distributions

(X,Y) has <u>joint density function</u> $f(x,y)$ if

$$P((X,Y)\in A) = \iint_A f(x,y)\,dx\,dy \text{, for any } A \subseteq \mathbb{R}^2$$

<u>Ex</u>: <u>Uniform</u> distribution on a disk



↳ So $f(x,y) = \begin{cases} c & \text{if } x^2+y^2 \le r^2 \\ 0, & \text{otherwise} \end{cases}$

$\to \iint_{x^2+y^2\le r^2} c\,dx\,dy = 1 \to c\cdot(\pi r^2) = 1$ or $c = \frac{1}{\pi r^2}$. Thus $f(x,y) = \begin{cases} \frac{1}{\pi r^2} \\ 0 \ldots \end{cases}$

Generally, given any compact set, we can define a uniform RV on it with $f(x,y) = \begin{cases} \frac{1}{\text{area}}, \ldots \\ 0, \ldots \end{cases}$

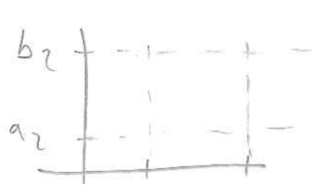
Ex 5.6 #28

Here Joint distribution function $F(x,y) = P(X\le x, Y \le y) = \iint_{x'\le x,\, y'\le y} f(x',y')\,dx'\,dy'$

Do Ex 5.6 #32

Generalizing FTC: $P(a < X \le b) = P(X\le b) - P(X\le a) = F(b) - F(a) = \int_a^b f(x)\,dx$

$$P(a_1 < X \le b_1,\ a_2 < Y \le b_2) = F(b_1,b_2) - F(b_1,a_2) - F(a_1,b_2) + F(a_1,a_2)$$



<u>Also</u>: $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x',y')\,dx'\,dy'$

$\Rightarrow \boxed{\frac{\partial^2 F}{\partial x \partial y} = f(x,y)}$ Do Ex 5.6 #33

Lecture 22 11/14/16

- Given joint density function $f(x,y)$, the marginal density of X is $f_X(x) = \int f(x,y)\,dy$
- (X,Y) independent if $f(x,y) = f_X(x) f_Y(y)$

Thm: If joint density $f(x,y)$ is seperable, ie $f(x,y) = g(x)h(y)$, then $\exists\, c>0$: $f_X(x) = c\,g(x)$ and $f_Y(y) = \frac{1}{c}h(y)$, and X, Y are independent.

5.6 #36 (a)
(b)

Here
The conditional density of X, given $Y=y$ is: $f_X(x \mid Y=y) = \dfrac{f(x,y)}{f_Y(y)}$ SKIP

5.6 #34

Chap 6 : Limit Thms

Sums of discrete RV's: $P(X+Y=z) = \sum_x P(X=x, Y=z-x)$ for any RV X,Y

$= \sum_x P(X=x)P(Y=z-x)$ for independent X,Y

Thm: Let $X \sim Bin(n,p)$ and $Y \sim Bin(m,p)$ with X and Y independent.
Then $X+Y \sim Bin(n+m,p)$

Pf: 1. $\rightsquigarrow$ $X+Y$ is RV for the # of successes in a sequence of $n+m$ (independent) trials, where each trial has prob. p of success
2. calculate using above formula, and identity $\sum_{j=0}^{k} C_{n,j}\, C_{m,k-j} = C_{n+m,k}$

Similarly,
Thm: Let $X \sim Pois(\lambda)$ and $Y \sim Pois(\mu)$ with X and Y independent.
Then $X+Y \sim Pois(\lambda+\mu)$

(Pf: Approximate $Pois(\lambda)$ by $Bin(\lfloor n\lambda \rfloor, \frac{1}{n})$ and use previous Thm)