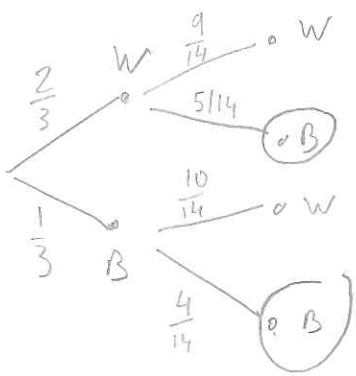
(Lecture 13 on 10/19/16 by D. Kostiuk (Conditional Prob.))

Lecture 14 on 10/21/16

3.5 #5
#8

3.2 Two-Stage Experiments

Ex: Urn has 15 balls, 10W, 5B.
Draw 2 balls.
Q: P(ball 2 is B)



P(ball 2 is B) = P(ball 1 and 2 are B)
+ P(ball 1 is W and ball 2 is B)

= P(ball 2 is B | ball 1 is B) P(ball 1 is B)
+ P(ball 2 is B | ball 1 is W) P(ball 1 is W)

$= \frac{4}{14} \cdot \frac{1}{3} + \frac{5}{14} \cdot \frac{2}{3}$

- $P(A) = P(A \cap (B \cup B^c)) = P(A \cap B) + P(A \cap B^c)$
$= P(A|B)P(B) + P(A|B^c)P(B^c)$

- More generally, suppose $\Omega = \bigcup_{i=1}^{n} B_i$ with $B_i \cap B_j = \phi$ for all $i \neq j$
(B_i's form partition for Ω)

Then $P(A) = P(A \cap \bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i) P(B_i)$

Law of Total Probability.

3.5 #29

Lecture 15   10/26/16

Redo problem 3.5 #5

N and S each draw 13 cards from a deck of 52.
S has 2 aces.

(a) $P(N \text{ has } 0 \text{ aces} \mid S \text{ has } 2 \text{ aces})$, (b) $P(N \text{ has } 1 \text{ ace} \mid S \text{ has } 2 \text{ aces})$, (c) $P(N \text{ has } 2 \text{ aces} \mid S \text{ has } 2 \text{ aces})$

First, we assume that S has drawn the 13 cards ~~first~~, and gotten 2 aces.

Then N draws 13 cards (as the 2nd player). Since we only care about the # of aces N will draw, we can assume that N has 52-13=39 cards to choose from, which contain 2 aces.

(a) $P(N \text{ has } 0 \text{ aces} \mid S \text{ has } 2 \text{ aces}) = \dfrac{C_{37,13}}{C_{39,13}} = 0.4386$

(b) $P(N \text{ has } 1 \text{ ace} \mid S \text{ has } 2 \text{ aces}) = \dfrac{C_{2,1}\, C_{37,12}}{C_{39,13}} = 0.4561$

(c) $P(N \text{ has } 2 \text{ aces} \mid S \text{ has } 2 \text{ aces}) = \dfrac{C_{2,2}\, C_{37,11}}{C_{39,13}} = 0.1053$

Now do problem 3.5 #4 (a HW problem)

Draw 5 cards from deck of 52. You get 2 spades and 3 hearts.

$P(1^{st} \text{ card drawn is spade} \mid 2 \text{ spades and } 3 \text{ hearts})$

A = "you draw 2 spades and 3 hearts"

B = "1st card drawn is spade"

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \mid B)\,P(B)}{P(A)} = \frac{\dfrac{C_{12,1}\,\cancel{C_{13,3}}}{C_{51,4}} \cdot \dfrac{13}{52}^{\,1/4}}{\dfrac{C_{13,2}\,\cancel{C_{13,3}}}{C_{52,5}}} = \frac{\overset{3}{\cancel{12}} \cdot \cancel{\tfrac{1}{4}} \cdot \dfrac{52 \cdot \cancel{51 \cdot 50 \cdot 49 \cdot 48}}{5 \cdot \cancel{4 \cdot 3 \cdot 2}}}{\dfrac{13 \cdot \overset{6}{\cancel{12}}}{\cancel{2}} \cdot \dfrac{\cancel{51 \cdot 50 \cdot 49 \cdot 48}}{\cancel{4 \cdot 3 \cdot 2}}}$$

$$= \frac{\cancel{3} \cdot \overset{4}{\cancel{52}}}{\cancel{13} \cdot \underset{2}{\cancel{6}} \cdot 5} = \frac{2}{5}$$

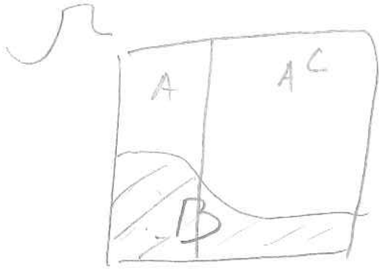
Bayes Formula

Example: 3.5 # 36

Binary digits (0's and 1's) sent over noisy channel.
Received as sent with prob. 0.9, but errors occur with prob 0.1
Assume 0's and 1's are sent equally likely
Q: Prob. that a 1 was sent, given a 1 was received.

A = "1 is sent", A^c = "0 is sent"
B = "1 is received", B^c = "0 is received"

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A^c)P(A^c)} = \frac{(0.9)(\frac{1}{2})}{(0.9)(\frac{1}{2})+(0.1)(\frac{1}{2})} = 0.9$$



Example: - 2% of people have a rare disease
- There is 95% chance of testing pos. for disease if you have it
- But there is a 10% chance of a false-positive test (i.e testing pos. but not having disease)

Q: P of having the disease, assuming you tested positive

A = "you have disease" A^c = "you don't have disease"
B = "you tested positive"

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A^c)P(A^c)} = \frac{(0.95)(0.02)}{(0.95)(0.02)+(0.1)(0.98)}$$

$$= \frac{0.019}{0.019+0.098} = \frac{0.019}{0.117} \approx 16\%$$, shockingly low!

Reason: This is rather high prob. i.e $P(B\cap A^c)$ or prob. of testing positive, even though you don't have the disease

Lecture 16/17 10/28/16 and 10/31/16

Continue Bayes' formula

3.5 #45

#54

#58

Joint Distributions

Relationship between 2 or more RV's X_1, X_2

3.5 #60

$Y \backslash X$	1	2	3	4
1	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
2	$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{12}$
3	$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{1}{12}$
4	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	0

$P(X=i, Y=j)$

$i, j \in \{1,2,3,4\}$

Note: Sum of all prob. in table adds to 1.

- Given joint distribution of (X,Y), how to recover distribution of X (or of Y)

→ Marginal Distribution: $P(X=x) = \sum_y P(X=x, Y=y)$

Ex (continued): $P(X=i) = 0 + 3 \cdot \frac{1}{12} = \frac{1}{4}$, $i = 1,2,3,4$

as expected

- Independent RV: $P(X=x, Y=y) = P(X=x)\,P(Y=y)$, $\forall x, y$.

In words: "Joint Distribution equals product of marginal distributions"

Ex (continued) Are X and Y independent?

No: $P(X=1, Y=1) = 0 \neq P(X=1)P(Y=1) = \left(\frac{1}{4}\right)^2$

Conditional Distribution

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$ often $P(Y=y) = \sum_x P(Y=y, X=x)$

Conditional Distribution of X, given that $Y=y$.

Ex (continued)

$$P(X=2 \mid Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/12}{1/4} = \frac{1}{3}$$

3.5 #64

$Y \backslash X$	0	3	6
1	0.3		
2	0.1	0.05	0.1

(a) $P(Y=2 \mid X=0) = \frac{1}{4}$ 3

(b) X, Y indep.

$$\frac{1}{4} = P(Y=2 \mid X=0) = \frac{P(X=0, Y=2)}{P(X=0)} = \frac{0.1}{P(X=0)} \to \boxed{P(X=0) = 0.4}$$

$$0.4 = P(X=0) = \sum_y P(X=0, Y=y) = P(X=0, Y=1) + P(X=0, Y=2)$$

or $0.4 = P(X=0, Y=1) + 0.1 \to \boxed{P(X=0, Y=1) = 0.3}$

By independence : $\frac{1}{4} = P(Y=2 \mid X=0) = \frac{P(X=0, Y=2)}{P(X=0)} = \frac{\cancel{P(X=0)} P(Y=2)}{\cancel{P(X=0)}}$

$$\to \boxed{P(Y=2) = \frac{1}{4}} \Rightarrow \boxed{P(Y=1) = \frac{3}{4}}$$

$$\frac{1}{4} = P(Y=2) = P(Y=2, X=0) + P(Y=2, X=3) + P(Y=2, X=6)$$

$$\to \frac{1}{4} = 0.1 + 0.05 + P(Y=2, X=6) \to \boxed{P(Y=2, X=6) = 0.1}$$

$$\begin{cases} P(X=3) = P(X=3, Y=1) + P(X=3, Y=2) \overset{\text{indep.}}{=} P(X=3) \cdot \frac{3}{4} + 0.05 \to \frac{1}{4} P(X=3) = 0.05 \text{ or } \boxed{P(X=3) = 0.2} \\ P(X=6) = P(X=6, Y=1) + P(X=6, Y=2) = P(X=6) \cdot \frac{3}{4} + 0.1 \to \frac{1}{4} P(X=6) = 0.1 \text{ or } \boxed{P(X=6) = 0.4} \end{cases}$$

$$\begin{cases} 0.2 = P(X=3) = P(X=3, Y=1) + 0.05 \to \\ 0.4 = P(X=6) = P(X=6, Y=1) + 0.1 \to \end{cases} \boxed{\begin{array}{l} P(X=3, Y=1) = 0.15 \\ P(X=6, Y=1) = 0.3 \end{array}}$$