Lecture 1

9/21/16

- Experiment with a certain number of outcomes

  Ex: Roll of die
  Flip of coin
  Flip of 2 coins
  etc

Ω = set of all possible outcomes, called Sample Space (may be finite, countably infinite, or uncountably infinite) eg $\mathbb{N}$ or $\mathbb{Z}$ — all 3 ex; eg $\mathbb{R}$

Ex: $\Omega_1 = \{1, 2, 3, 4, 5, 6\}$
$\Omega_2 = \{H, T\}$
$\Omega_3 = \{(H,H), (H,T), (T,H), (T,T)\}$
etc

- Events: Subsets of sample space → includes empty event $\emptyset$

  Ex: $\Omega_1 \to A$ = "roll of die is 5 or less" $= \{1,2,3,4,5\}$
  $\Omega_2 \to B$ = "flip of 2 coins giving different values" $= \{(H,T), (T,H)\}$

- Events occur with certain probabilities

  Ex: $\Omega_1 \to P(A) = \frac{5}{6}$
  $\Omega_2 \to P(B) = \frac{2}{4} = \frac{1}{2}$

- Generally, one must prescribe a probability function P, which assigns to each event a number in $[0,1]$

P satisfies axioms:
(1) For all A: $0 \le P(A) \le 1$
(2) $P(\Omega) = 1$
(3) If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$
(4) If A_i, $i = 1, 2, \dots$, is a sequence of pairwise disjoint events (ie $A_i \cap A_j = \emptyset$ if $i \neq j$), then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Note: $A \cap B \to$ event A and B
$A \cup B \to$ event A or B
$A^c \to$ complement of event A

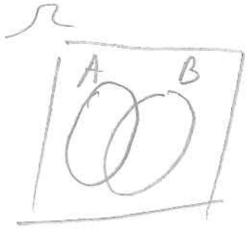
<u>Properties</u>: 1. $P(A^c) = 1 - P(A)$

$\begin{cases} A \cap A^c = \phi \\ A \cup A^c = \Omega \end{cases}$

$\overset{(iii)}{\Longrightarrow} 1 = P(\Omega) = P(A) + P(A^c)$

2. Inclusion-Exclusion Principle: $P(A \cup B) = \underbrace{P(A) + P(B)}_{\text{Inclusion}} \underbrace{- P(A \cap B)}_{\text{Exclusion}}$,

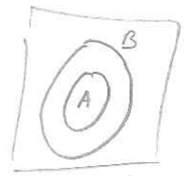
Inclusion ↓ counts intersection twice



Also valid for cardinality (ie number of elements) of sets.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

3. Monotonicity: $A \subseteq B \Rightarrow P(A) \leq P(B)$



$\begin{cases} A \cap (B \cap A^c) = \phi \\ A \cup (B \cap A^c) = B \end{cases}$

$\overset{(iii)}{\Longrightarrow} P(B) = P(A) + \underbrace{P(B \cap A^c)}_{\geq 0 \text{ by (i)}}$

4. Monotone limits: If $\begin{cases} A_i \uparrow A \ (\text{ie } A_1 \subseteq A_2 \subseteq A_3 \cdots \text{ and } \bigcup_{i=1}^{\infty} A_i = A), \\ B_i \downarrow B \ (\text{ie } B_1 \supseteq B_2 \supseteq B_3 \cdots \text{ and } \bigcap_{i=1}^{\infty} B_i = B), \end{cases}$ then $\begin{cases} \lim_{i \to \infty} P(A_i) = P(A) \\ \lim_{i \to \infty} P(B_i) = P(B) \end{cases}$

proof uses (iv)

Do problem 1.7 #4 (sample space, events, prob.)

#14 (inclusion-exclusion)

Birthday problem

Do problems at end of Lecture 1. (1.7 #4, #14)

Birthday problem: Class of N students

(Ignore Feb 29) Q: $P($ at least 2 students have same birthday$)$

Ω = set of outcomes, ie 1 element is 1 possible sequence of N birthdays

$|\Omega| = 365^N$

Now $|\{\text{no 2 students have same birthday}\}| = 365 \cdot 364 \cdots (365-(N-1))$

$\Rightarrow P(\{\text{no 2 students have same birthday}\}) = \dfrac{365 \cdot 364 \cdots (365-(N-1))}{365^N}$

As of 9/23/16, there are 27 students in this class

p. 9

$\Rightarrow P(\{\text{no 2 students have same birthday}\}) = 0.37$

$\Rightarrow P(\{\text{at least 2 students have same birthday}\}) = 0.67$

Also, according to this table, to have $P(\{\text{at least 2 students have same birthday}\}) > 50\%$,

you need $N \geq 23$.

Independence

Def: A, B are independent if $P(A \cap B) = P(A)\,P(B)$

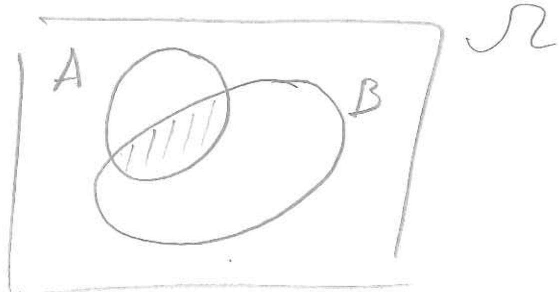
Main idea: If A occurs, it should not have impact on B occurring or not (+ vice versa)

How to see this from the definition above?

New concept: Conditional Probability

Assume. $P(B)>0$ $\quad P(A|B)$ = Prob. that A occurs, given that B occured $= \frac{P(A\cap B)}{P(B)}$

Notice: By saying "given that B occured", we have changed the sample space to B.



Link to independence. Suppose A, B indep. Then $P(A|B) = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A)$, i.e. Prob that A occurs is same as prob that A occurs, given that B has occurred

Do Problem 1.7 #21
#23
#25

Generalizing independence to more than 2 events

1. $A_1, \ldots, A_n$ are pairwise independent if (A_i, A_j) are independent for all $i \neq j$, ie $P(A_i \cap A_j) = P(A_i)P(A_j)$

2. $A_1, \ldots, A_n$ are independent if for all $k = 2, \ldots, n$ and for all $1 \le i_1 < i_2 < \ldots < i_k \le n$:

$$P(A_{i_1} \cap A_{i_2} \ldots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2})\ldots P(A_{i_k})$$

Remark: Pairwise independence $\not\Rightarrow$ Independence

<u>Ex</u>: Roll 3 dice $\to |\Omega| = 6^3$

A = "1st, 2nd are same", B = "1st, 3rd are same", C = "2nd, 3rd are same"

$|A| = 6^2 = |B| = |C|$ so $P(A) = P(B) = P(C) = \frac{6^2}{6^3} = \frac{1}{6}$

$P(A \cap B) = P(\text{"1st, 2nd, and 3rd are same"}) = \frac{6}{6^3} = \frac{1}{6^2} = P(A)P(B) = \frac{1}{6} \cdot \frac{1}{6}$

and same for $P(A \cap C)$, $P(B \cap C)$

But $P(A \cap B \cap C) = P(A \cap B) = \frac{1}{6^2} \neq P(A)P(B)P(C) = \frac{1}{6^3}$

Random Variables (RV)

Def: A number associated to the outcome of an experiment

Ex:
1. Σ in roll of 3 dice
2. # of H in flip of N coins
3. # of times one should roll a die until a 6 shows.

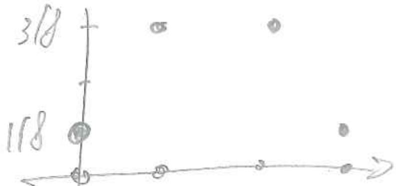
All are discrete RV (as opposed to Cont. RV, which are considered later)

i.e. number takes finitely many, or uncountably many values

Distribution of a RV is $P(X=x)$, the prob. that RV X takes any of its values x.

Ex: 2. $X = \#$ of H in flip of 3 coins

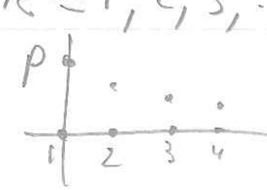
	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
	TTT	HTT, THT, TTH	HHT, HTH, THH	HHH



3. An example of the geometric RV

# of times an experiment with prob of success p has to be repeated until 1st success.

$$P(X=k) = \underbrace{(1-p)^{k-1}}_{n-1 \text{ failures}} \underbrace{p}_{1^{st} \text{ success}} \quad , k = 1, 2, 3, \ldots$$



in ex.: $p = 1/6$

Remark: Recall $P(\Omega)=1$

$$\overset{?}{\Rightarrow} \sum_{k=1}^{\infty} P(X=k) = 1 \,?$$

$$\sum_{k=1}^{\infty} P(X=k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{n=0}^{\infty} (1-p)^n = p \cdot \frac{1}{1-(1-p)} = p \cdot \frac{1}{p} = 1 \checkmark$$

Ex: 1.7 # 35
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Birthday Problem II: How large should class size be such that there is someone having your birthday with prob. 0.5 or more.

$$P\left(\begin{array}{c}\text{nobody has your birthday} \\ \text{in class of size } n\end{array}\right) = \left(\frac{364}{365}\right)^n \le \frac{1}{2}$$

$$\Rightarrow n \ln\left(\frac{364}{365}\right) \le -\ln(2) \overset{(\text{neg. \#})}{\Rightarrow} n \ge 252.7$$

ie $n = 253$ needed!

Expected Value: Let X be a RV taking values x with Prob. $P(X=x)$

$$E(X) = \sum_x x \, P(X=x)$$

Ex: Geometric

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Expected Value of RV X: Given Distribution $P(X=x)$

$$E(x) = \sum_x x P(X=x)$$

Ex: Tetrahedral die with sides 1, 3, 4, 9

$$E(x) = 1\cdot\frac{1}{4} + 3\cdot\frac{1}{4} + 4\cdot\frac{1}{4} + 9\cdot\frac{1}{4} = \frac{17}{4}$$

Ex: Geometric $P(x=k) = (1-p)^{k-1}p$, $k=1,2,\dots$

$$E(x) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p\cdot\frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$

Recall: $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, if $|x|<1$

$\xrightarrow{\frac{d}{dx}}$ $\sum_{k=0}^{\infty} k x^{k-1} = -\left(-\frac{1}{(1-x)^2}\right) = \frac{1}{(1-x)^2}$, if $|x|<1$

$\sum_{k=1}^{\infty} k x^{k-1}$ "

Property (proved later): If $x_1,\dots,x_n$ are RV, then $E(x_1+\dots+x_n) = E(x_1)+\dots+E(x_n)$

Ex 1.7 #40

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Moments and Variance

Property: Let X be discrete RV, and $f: S \to \mathbb{R}$
taking values in S and $P(X=x)$ dist

Then $Y = f(X)$ is also a discrete RV with $E(Y) = \sum_{x \text{ in } S} f(x) P(X=x)$

Ex: $f(x) = x^k$ yields k^{th} moments

e.g. 3^{rd} moment of X: $E(X^3)$

Property: $E(aX+b) = \sum_{x \text{ in } S} (ax+b) P(X=x)$ $\quad a, b$ constants

$= a \sum_{x \text{ in } S} x P(X=x) + b \sum P(X=x)$

$= a E(X) + b$

Variance of X: If $E(X^2) < \infty$, then variance of X is

$$Var(X) = E\left(\left(X - E(X)\right)^2\right) \overset{Prop}{=} E(X^2) - \left(E(X)\right)^2$$

Note: $\sigma(X) =$ st. dev. of $X := \sqrt{Var(X)}$

Ex 1.7 #52