6.7 Exercises

Expected value of sums

1. A man plays roulette and bets \$1 on black 19 times. He wins \$1 with probability 18/38 and loses \$1 with probability 20/38. What are his expected winnings?

2. Suppose we draw 13 cards out of a deck of 52. What is the expected value of the number of aces we get?

3. Suppose we pick 3 students at random from a class with 10 boys and 15 girls. Let X be the number of boys selected and Y be the number of girls selected. Find $E(X - Y)$.

4. 12 ducks fly overhead. Each of 6 hunters picks one duck at random to aim at and kills it with probability 0.6. (a) What is the mean number of ducks that are killed? (b) What is the expected number of hunters who hit the duck they aim at?

5. 10 people get on an elevator on the first floor of a seven-story building. Each gets off at one of the six higher floors chosen at random. What is the expected number of stops the elevator makes?

6. Suppose Noah started with n pairs of animals on the ark and m of them died. If we suppose that fate chose the m animals at random, what is the expected number of complete pairs that are left? Find the answer when $n = 100$ and $m = 10$.

7. Suppose we draw 5 cards out of a deck of 52. What is the expected number of different suits in our hand? For example, if we draw K♠ 3♠ 10♡ 8♡ 6♣, there are three different suits in our hand.

8. Suppose we draw cards out of a deck without replacement. How many cards do we expect to draw out before we get an ace?

9. A calculus class has 150 students. Assume there are 365 days in a year and note that $150/365 = 0.410959$. (a) What is the probability that at least one student is born on April 1? (b) Let N be the number of days on which at least one student has a birthday. Find the expected value EN.

Variance and covariance

10. Roll two dice and let $Z = XY$ be the product of the two numbers obtained. What is the mean and variance of Z?

11. Suppose X and Y are independent with $EX = 1$, $EY = 2$, $\text{var}(X) = 3$, and $\text{var}(Y) = 1$. Find the mean and variance of $3X + 4Y - 5$.

12. In a class with 18 boys and 12 girls, boys have probability 1/3 of knowing the answer and girls have probability 1/2 of knowing the answer to a typical question the teacher asks. Assuming that whether or not the students know the answer are independent events, find the mean and variance of the number of students who know the answer.

13. At a local high school, 12 boys and 4 girls are applying to MIT. Suppose that the boys have a 10% chance of getting in and the girls have a 20% chance. (a) Find the mean and variance of the number of students accepted. (b) What is more likely: 2 boys and no girls accepted or 1 boy and 1 girl?

14. Let N_k be the number of independent trials we need to get k successes when success has probability p. Find the mean and variance of N_k.

15. Suppose we roll a die repeatedly until we see each number at least once and let R be the number of rolls required. Find the mean and variance of R.

16. Suppose X takes on the values $-2, -1, 0, 1, 2$ with probability 1/5 each, and let $Y = X^2$. (a) Find $\text{cov}(X, Y)$. (b) Are X and Y independent?

Chebyshev's inequality

17. Suppose that it is known that the number of items produced at a factory per week is a random variable X with mean 50. (a) What can we say about the probability $X \geq 75$? (b) Suppose that the variance of X is 25. What can we say about $P(40 < X < 60)$?

18. Let $X = \text{binomial}(4, 1/2)$. Use Chebyshev's inequality to estimate $P(|X - 2| \geq 2)$ and compare with the exact probability.

19. Let $\bar{X}_{10,000}$ be the fraction of heads in 10,000 tosses. Use Chebyshev's inequality to bound $P(|\bar{X}_n - 1/2| \geq 0.01)$ and the normal approximation to estimate this probability.

20. Let X have a Poisson distribution with mean 16. Estimate $P(X \geq 28)$ using (a) Chebyshev's inequality and (b) the normal approximation.

Central limit theorem, I. Coin flips

21. A person bets you that in 100 tosses of a fair coin the number of heads will differ from 50 by 3 or fewer, that is, he will win if the difference is 4 or more. What is the probability that he will win this bet?

22. Suppose we toss a coin 100 times. Which is bigger, the probability of exactly 50 heads or at least 60 heads?

23. Bill is a student at Cornell. In any given course he gets an A with probability 1/2 and a B with probability 1/2. Suppose the outcomes of his courses are independent. In his 4 years at Cornell he will take 33 courses. If he can get 22 A's and only 11 B's he can graduate with a 3.666 average. What is the probability that he will do this?

24. In a 162-game season find the approximate probability that a team with a 0.5 chance of winning will win at least 87 games.

25. British Airways and United offer identical service on two flights from New York to London that leave at the same time. Suppose that they are competing for the same pool of 400 customers who choose an airline at random. What is the probability that United will have more customers than its 230 seats?

26. A probability class has 30 students. As part of an assignment, each student tosses a coin 200 times and records the number of heads. What is the probability that no student gets exactly 100 heads?

27. A fair coin is tossed 2,500 times. Find a number m so that the chance that the number of heads is between 1,250 − m and 1,250 + m is approximately 2/3.

CLT, II. Biased coins

28. Suppose we roll a die 600 times. What is the approximate probability that the number of 1's obtained lies between 90 and 110?

29. Suppose that each of 300 patients has a probability of 1/3 of being helped by a treatment. Find approximately the probability that more than 120 patients are helped by the treatment.

30. Suppose that 10% of a certain brand of jelly beans are red. Use the normal approximation to estimate the probability that in a bag of 400 jelly beans there are at least 45 red ones.

31. A basketball player makes 80% of his free throws on the average. Use the normal approximation to compute the probability that in 25 attempts he will make at least 23.

32. Suppose that we roll two dice 180 times and we are interested in the probability that we get exactly 5 double sixes. Find (a) the normal approximation, (b) the exact answer, and (c) the Poisson approximation.

33. A gymnast has a difficult trick with a 10% chance of success. She tries the trick 25 times and wants to know the probability that she will get exactly two successes. Compute the (a) exact answer, (b) Poisson approximation, and (c) normal approximation.

34. A student is taking a true/false test with 48 questions. (a) Suppose she has a probability $p = 3/4$ of getting each question right. What is the probability that she will get at least 38 right? (b) Answer the last question if she knows the answers to half the questions and flips a coin to answer the other half. Notice that in each case the expected number of questions she gets right is 36.

35. To estimate the percent of voters who oppose a certain ballot measure, a survey organization takes a random sample of 200 voters. If 45% of the voters oppose the measure, estimate the chance that (a) exactly 90 voters in the sample oppose the measure and (b) more than half the voters in the sample oppose the measure.

36. An airline knows that in the long run only 90% of passengers who book a seat show up for their flight. On a particular flight with 300 seats there are 324 reservations. Assuming passengers make independent decisions what is the chance that the flight will be overbooked?

37. Suppose that 15% of people don't show up for a flight, and suppose that their decisions are independent. How many tickets can you sell for a plane with 144 seats and be 99% sure that not too many people will show up?

38. A seed manufacturer sells seeds in packets of 50. Assume that each seed germinates with probability 0.99 independently of all the others. The manufacturer promises to replace, at no cost to the buyer, any packet with 3 or more seeds that do not germinate. (a) Use the Poisson to estimate the probability a packet must be replaced. (b) Use the normal to estimate the probability that the manufacturer has to replace more than 70 of the last 4,000 packets sold.

39. An electronics company produces devices that work properly 95% of the time. The new devices are shipped in boxes of 400. The company wants to guarantee that k or more devices per box work. What is the largest k so that at least 95% of the boxes meet the warranty?

CLT, III. General distributions

40. The number of students who enroll in a psychology class is Poisson with mean 100. If the enrollment is > 120, then the class will be split into two sections. Estimate the probability that this will occur.

41. On each bet a gambler loses \$1 with probability 0.7, loses \$2 with probability 0.2, and wins \$10 with probability 0.1. Estimate the probability that the gambler's winnings will be ≥ 0 after 100 bets.

42. Suppose we roll a die 10 times. What is the approximate probability that the sum of the numbers obtained lies between 30 and 40?

43. Members of the Beta Upsilon Zeta fraternity each drink a random number of beers with mean 6 and standard deviation 3. If there are 81 fraternity members, how much should they buy so that using the normal approximation they are 93.32% sure they will not run out?

44. An insurance company has 10,000 automobile policy holders. The expected yearly claim per policy holder is \$240 with a standard deviation of \$800. Approximate the probability that the yearly claim exceeds \$2.7 million.

45. A die is rolled repeatedly until the sum of the numbers obtained is larger than 200. What is the probability that you need more than 66 rolls to do this?

46. Suppose that the checkout time at a grocery store has a mean of 5 minutes and a standard deviation of 2 minutes. Estimate the probability that a checker will serve at least 49 customers during her 4-hour shift.

Confidence intervals

47. Of the first 10,000 votes cast in an election, 5,180 were for candidate A. Find a 95% confidence interval for the fraction of votes that candidate A will receive.

48. A bank examines the records of 150 patrons and finds that 63% have savings accounts. Find a 95% confidence interval for the fraction of people with savings accounts.

49. Among 625 randomly chosen Swedish citizens, it was found that 25 had previously been citizens of another country. Find a 95% confidence interval for the true proportion.

50. A sample of 2,089 handheld video games revealed that 212 broke within the first 3 months of operation. Find a 95% confidence interval for the true proportion that break in the first 3 months.

51. Suppose we take a poll of 2,500 people. What percentage should the leader have for us to be 99% confident that the leader will be the winner?

52. For a class project, you are supposed to take a poll to forecast the outcome of an election. How many people do you have to ask so that with probability 0.95 your estimate will not differ from the true outcome by more than 5%?

Hypothesis testing

We will use casual language to state these problems. The precise formulation of the first one is if $p = 1/6$, then what is the probability that we will observe 3,123 or more sixes in 18,000 rolls. Here, and in what we follows, we will ignore the fact that we should multiply by 2 to get a more accurate idea of how odd the observation is.

53. A casino owner is concerned based on past experience that his dice show 6 too often. He makes his employees roll a die 18,000 times and they observe 3,123 sixes. Is the die biased?

54. We suspect that a bridge player is cheating by putting an honor card (ace, king, queen, or jack) at the bottom of the deck when he shuffles so that this card will end up in his hand. In 16 times when he dealt, the last card dealt to him was an honor on 9 occasions. Are we confident that he is cheating?

55. If both parents carry one dominant (A) and one recessive gene (a) for a trait then Mendelian inheritance predicts that 1/4 of the offspring will have both recessive genes (aa) and show the recessive trait. If among 96 offspring of Aa parents we find 30 are aa, is this consistent with Mendelian inheritance?

56. A softball player brags that he is a 0.300 hitter, yet at the end of the season he has gotten 21 hits in 84 at bats. Is this just bad luck?

57. In a 60-day period in Ithaca 12 days were rainy. Is this observation consistent with the belief that the true proportion of rainy days is 1/3?

58. In a poll of 900 Americans in 1978, 65% said that extramarital sex was wrong, whereas a similar poll in 1985 found that 72% had the same opinion. Are we confident that opinions have changed?

59. A psychic claims to be able to guess the suit of a card without seeing it. In 52 attempts, someone who is just guessing will get 13 right on the average. How many would he have to get right so that we are about 99.88% sure he is not guessing.