Substituting in the analogous continuous quantities, we have

$$f(x, y) = f_X(x) f_Y(y|X = x) \tag{5.16}$$

The next example demonstrates the use of (5.16) to compute a joint distribution.

**Example 5.29** Suppose we pick a point uniformly distributed on (0, 1), call it $X$, and then pick a point $Y$ uniformly distributed on $(0, X)$.

To find the joint density of $(X, Y)$ we note that

$$\begin{aligned} f_X(x) &= 1 && \text{for } 0 < x < 1 \\ f_Y(y|X = x) &= 1/x && \text{for } 0 < y < x \end{aligned}$$

So using (5.16), we have

$$f(x, y) = f_X(x) f_Y(y|X = x) = 1/x \quad \text{for } 0 < y < x < 1$$

To complete the picture we compute

$$f_Y(y) = \int f(x, y)\, dx = \int_y^1 \frac{1}{x}\, dx = -\ln y$$

$$f_X(x|Y = y) = \frac{f(x, y)}{f_Y(y)} = \frac{1/x}{-\ln y} \quad \text{for } y < x < 1$$

Again the conditional density of $X$ given $Y = y$ is obtained by fixing $y$, regarding the joint density function as a function of $x$, and then normalizing so that the integral is 1. The reader should note that although $X$ is uniform on (0, 1) and $Y$ is uniform on $(0, X)$, $X$ is not uniform on $(Y, 1)$ but has a greater probability of being near $Y$.

## 5.6 Exercises

### Density functions

1. Suppose $X$ has density function $f(x) = c(3 - |x|)$ when $-3 < x < 3$. What value of $c$ makes this a density function?

2. Consider $f(x) = c(1 - x^2)$ for $-1 < x < 1$ and 0 otherwise. What value of $c$ should we take to make $f$ a density function?

3. Suppose $X$ has density function $6x(1 - x)$ for $0 < x < 1$ and 0 otherwise. Find (a) $EX$, (b) $E(X^2)$, and (c) $\text{var}(X)$.

4. Suppose $X$ has density function $x^2/9$ for $0 < x < 3$ and 0 otherwise. Find (a) $EX$, (b) $E(X^2)$, and (c) $\text{var}(X)$.

5. Suppose $X$ has density function $x^{-2/3}/21$ for $1 < x < 8$ and 0 otherwise. Find (a) $EX$, (b) $E(X^2)$, and (c) $\text{var}(X)$.

## Distribution functions

6. $F(x) = 3x^2 - 2x^3$ for $0 < x < 1$ (with $F(x) = 0$ if $x \le 0$ and $F(x) = 1$ if $x \ge 1$) defines a distribution function. Find the corresponding density function.

7. Let $F(x) = e^{-1/x}$ for $x > 0$ and $F(x) = 0$ for $x \le 0$. Is $F$ a distribution function? If so, find its density function.

8. Let $F(x) = 3x - 2x^2$ for $0 \le x \le 1$, $F(x) = 0$ for $x \le 0$, and $F(x) = 1$ for $x \ge 1$. Is $F$ a distribution function? If so, find its density function.

9. Suppose $X$ has density function $f(x) = x/2$ for $0 < x < 2$ and 0 otherwise. Find (a) the distribution function, (b) $P(X < 1)$, (c) $P(X > 3/2)$, and (d) the median.

10. Suppose $X$ has density function $f(x) = 4x^3$ for $0 < x < 1$ and 0 otherwise. Find (a) the distribution function, (b) $P(X < 1/2)$, (c) $P(1/3 < X < 2/3)$, and (d) the median.

11. Suppose $X$ has density function $x^{-1/2}/2$ for $0 < x < 1$ and 0 otherwise. Find (a) the distribution function, (b) $P(X > 3/4)$, (c) $P(1/9 < X < 1/4)$, and (d) the median.

12. Suppose $P(X = x) = x/21$ for $x = 1, 2, 3, 4, 5, 6$. Find all the medians of this distribution.

13. Suppose $X$ has a Poisson distribution with $\lambda = \ln 2$. Find all the medians of $X$.

14. Suppose $X$ has a geometric distribution with success probability 1/4; that is, $P(X = k) = (3/4)^{k-1}(1/4)$. Find all the medians of $X$.

15. Suppose $X$ has density function $3x^{-4}$ for $x \ge 1$. (a) Find a function $g$ so that $g(X)$ is uniform on $(0, 1)$. (b) Find a function $h$ so that if $U$ is uniform on $(0, 1)$, $h(U)$ has density function $3x^{-4}$ for $x \ge 1$.

16. Suppose $X_1, \ldots, X_n$ are independent and have distribution function $F(x)$. Find the distribution functions of (a) $Y = \max\{X_1, \ldots, X_n\}$ and (b) $Z = \min\{X_1, \ldots, X_n\}$

17. Suppose $X_1, \ldots, X_n$ are independent exponential($\lambda$). Show that

$$\min\{X_1, \ldots, X_n\} = \text{exponential}(n\lambda)$$

## Functions of random variables

18. Suppose $X$ has density function $f(x)$ for $a \le x \le b$ and $Y = cX + d$, where $c > 0$. Find the density function of $Y$.

19. Show that if $X = \text{exponential}(1)$, then $Y = X/\lambda$ is exponential($\lambda$).

20. Suppose $X$ is uniform on $(0, 1)$. Find the density function of $Y = X^n$.

21. Suppose $X$ has density $x^{-2}$ for $x \ge 1$ and $Y = X^{-2}$. Find the density function of $Y$.

22. Suppose $X$ has an exponential distribution with parameter $\lambda$ and $Y = X^{1/\alpha}$. Find the density function of $Y$. This is the *Weibull distribution.*

23. Suppose $X$ has an exponential distribution with parameter 1 and $Y = \ln(X)$. Find the distribution of $Y$. This is the *double exponential distribution.*

24. Suppose $X$ is uniform on $(0, \pi/2)$ and $Y = \sin X$. Find the density function of $Y$. The answer is called the *arcsine law* because the distribution function contains the arcsine function.

25. Suppose $X$ has density function $f(x)$ for $-1 \le x \le 1$ and 0 otherwise. Find the density function of (a) $Y = |X|$ and (b) $Z = X^2$.

26. Suppose $X$ has density function $x/2$ for $0 < x < 2$ and 0 otherwise. Find the density function of $Y = X(2 - X)$ by computing $P(Y \ge y)$ and then differentiating.

## Joint distributions

27. Suppose $X$ and $Y$ have joint density $f(x, y) = c(x + y)$ for $0 < x, y < 1$. (a) What is $c$? (b) What is $P(X < 1/2)$?

28. Suppose $X$ and $Y$ have joint density $f(x, y) = 6xy^2$ for $0 < x, y < 1$. What is $P(X + Y < 1)$?

29. Suppose $X$ and $Y$ have joint density $f(x, y) = 2$ for $0 < y < x < 1$. Find $P(X - Y > z)$.

30. Suppose $X$ and $Y$ have joint density $f(x, y) = 1$ for $0 < x, y < 1$. Find $P(XY \le z)$.

31. Two people agree to meet for a drink after work but they are impatient and each will wait only 15 minutes for the other person to show up. Suppose that they each arrive at independent random times uniformly distributed between 5 P.M. and 6 P.M. What is the probability they will meet?

32. Suppose $X$ and $Y$ have joint density $f(x, y) = e^{-(x+y)}$ for $x, y > 0$. Find the distribution function.

33. Suppose $X$ is uniform on $(0, 1)$ and $Y = X$. Find the joint distribution function of $X$ and $Y$.

34. A pair of random variables $X$ and $Y$ takes values between 0 and 1 and has $P(X \le x, Y \le y) = x^3y^2$ when $0 \le x, y \le 1$. Find the joint density function.

35. Given the joint distribution function $F_{X,Y}(x, y) = P(X \le x, Y \le y)$, how do you recover the marginal distribution function $F_X(x) = P(X \le x)$?

36. Suppose $X$ and $Y$ have joint density $f(x, y)$. Are $X$ and $Y$ independent if

(a) $f(x, y) = xe^{-x(1+y)}$ for $x, y \ge 0$?
(b) $f(x, y) = 6xy^2$ when $x, y \ge 0$ and $x + y \le 1$?
(c) $f(x, y) = 2xy + x$ when $0 < x < 1$ and $0 < y < 1$?
(d) $f(x, y) = (x + y)^2 - (x - y)^2$ when $0 < x < 1$ and $0 < y < 1$?

In each case, $f(x, y) = 0$ otherwise.

37. Suppose a point $(X, Y)$ is chosen at random from the disk $x^2 + y^2 \le 1$. Find (a) the marginal density of $X$ and (b) the conditional density of $Y$ given $X = x$.

38. Suppose $X$ and $Y$ have joint density $f(x, y) = x + 2y^3$ when $0 < x < 1$ and $0 < y < 1$. (a) Find the marginal densities of $X$ and $Y$. (b) Are $X$ and $Y$ independent?

39. Suppose $X$ and $Y$ have joint density $f(x, y) = 6y$ when $x > 0$, $y > 0$, and $x + y < 1$. (a) Find the marginal densities of $X$ and $Y$ and (b) the conditional density of $X$ given $Y = y$.

40. Suppose $X$ and $Y$ have joint density $f(x, y) = 10x^2 y$ when $0 < y < x < 1$. (a) Find the marginal densities of $X$ and $Y$ and (b) the conditional density of $Y$ given $X = x$.