

## 3.5 Exercises

The conditional expectation is mean of the conditional distribution:

$$E(X|Y = y) = \sum_x xP(X = x|Y = y)$$

In the previous example,

$$E(X|Y = 4) = (5(0.10) + 4(0.15) + 3(0.10))/0.35 = 4$$

$$E(X|Y = 3) = (5(0.05) + 4(0.15) + 3(0.15) + 2(0.05))/0.40 = 3.5$$

$$E(X|Y = 2) = (3(0.10) + 2(0.10) + 1(0.05))/0.25 = 2.2$$

**Example 3.30**

**Simpson's paradox** is the phenomenon that means of subgroups can show much different patterns than the mean of the group as a whole. For a real-life example, consider the average SAT verbal score. It was 504 in 1981 and 21 years later in 2002 it was again 504. However, when we break things down by ethnic groups, we see that all of them increased their scores:

	1981	2002
Non-Hispanic whites	519	527
African-Americans	412	431
Mexican-Americans	438	446
Asian-Americans	474	501

The explanation is simple: minorities made up a much larger portion of the testing population in 2002 than in 1981, and although they have shown significant improvement their averages are lower than non-Hispanic whites, which reduces the overall mean.

**3.5 Exercises****Conditional probability**

1. A friend flips two coins and tells you that at least one is head. Given this information, what is the probability that the first coin is head?
2. A friend rolls two dice and tells you that there is at least one 6. What is the probability the sum is at least 9?
3. Suppose we roll two dice. What is the probability that the sum is 7 given that neither die showed a 6?
4. Suppose you draw 5 cards out of a deck of 52 and get 2 spades and 3 hearts. What is the probability that the first card drawn was a spade?

## Conditional Probability

5. Two people, whom we call South and North, draw 13 cards out of a deck of 52. South has two aces. What is the probability that North has (a) none? (b) One? (c) The other two?
6. An urn contains 8 red, 7 blue, and 5 green balls. You draw out two balls and they are different colors. Given this, what is the probability that the two balls were red and blue?
7. Suppose 60% of the people subscribe to newspaper A, 40% to newspaper B, and 30% to both. If we pick a person at random who subscribes to at least one newspaper, what is the probability that she subscribes to newspaper A?
8. In a town 40% of families have a dog and 30% have a cat. 25% of families with a dog also have a cat. (a) What fraction of people have a dog or cat? (b) What is the probability that a family with a cat has a dog?
9. Plumber Bob does 40% of the plumbing jobs in a small town. 30% of the people in town are unhappy with their plumbers, but 50% of Bob's customers are unhappy with his work. If your neighbor is not happy with his plumber, what is the probability that it was Bob?
10. An ectopic pregnancy is twice as likely if a woman smokes cigarettes. If 25% of women of childbearing age are smokers, what fraction of ectopic pregnancies occur to smokers?
11. Brown eyes are dominant over blue. That is, there are two alleles  $B$  and  $b$ .  $bb$  individuals have blue eyes but other combinations have brown eyes. Your parents and you have brown eyes but your brother has blue. So you can infer that both of your parents are heterozygotes, that is, have genetic type  $Bb$ . Given this information what is the probability that you are a homozygote.
12. Suppose that the probability that a married man votes is 0.45, the probability a married woman votes is 0.4, and the probability a woman votes given that her husband does is 0.6. What is the probability that (a) both vote? (b) A man votes given that his wife does?
13. Two events have  $P(A) = 1/4$ ,  $P(B|A) = 1/2$ , and  $P(A|B) = 1/3$ . Compute  $P(A \cap B)$ ,  $P(B)$ , and  $P(A \cup B)$ .
14.  $A$ ,  $B$ , and  $C$  are events with  $P(A) = 0.3$ ,  $P(B) = 0.4$ , and  $P(C) = 0.5$ ,  $A$  and  $B$  are disjoint,  $A$  and  $C$  are independent, and  $P(B|C) = 0.1$ . Find  $P(A \cup B \cup C)$ .

**Two-stage experiments**

15. From a signpost that says MIAMI two letters fall off. A friendly drunk puts the two letters back into the two empty slots at random. What is the probability that the sign still says MIAMI?
16. Two balls are drawn from an urn with balls numbered from 1 up to 10. What is the probability that the two numbers will differ by more ( $>$ ) than 3?
17. How can 5 black and 5 white balls be put into two urns to maximize the probability that a white ball is drawn when we draw from a randomly chosen urn?
18. Suppose we draw  $k$  cards out of a deck. What is the probability that we do not draw an ace? Is the answer larger or smaller than  $(3/4)^k$ ?
19. You and a friend each roll two dice. What is the probability that you both have the same two numbers?
20. In a dice game the "dealer" rolls two dice, the player rolls two dice, and the player wins if his total is larger than the dealer's. What is the probability that the player wins?
21. What is the most likely total for the sum of four dice and what is its probability?
22. Charlie draws 5 cards out of a deck of 52. If he gets at least three of one suit, he discards the cards not of that suit and then draws until he again has 5 cards. For example, if he gets 3 hearts, 1 club, and 1 spade, he throws the 2 nonhearts away and draws 2 more. What is the probability that he will end up with 5 cards of the same suit?
23. Suppose 60% of the people in a town will get exposed to flu in the next month. If you are exposed and not inoculated then the probability of your getting the flu is 80%, but if you are inoculated that probability drops to 15%. Of two executives at Beta Company, one is inoculated and one is not. What is the probability at least one will not get the flu? Assume that the events that determine whether or not they get the flu are independent.
24. John takes the bus with probability 0.3 and the subway with probability 0.7. He is late 40% of the time when he takes the bus, but only 20% of the time when he takes the subway. What is the probability that he is late for work?
25. The population of Cyprus is 70% Greek and 30% Turkish. 20% of the Greeks and 10% of the Turks speak English. What fraction of the people of Cyprus speak English?

## Conditional Probability

26. You are going to meet a friend at the airport. Your experience tells you that the plane is late 70% of the time when it rains, but is late only 20% of the time when it does not rain. The weather forecast that morning calls for a 40% chance of rain. What is the probability that the plane will be late?

27. Two boys have identical piggy banks. The older boy has 18 quarters and 12 dimes in his; the younger boy, 2 quarters and 8 dimes. One day the two banks get mixed up. You pick up a bank at random and shake it until a coin comes out. What is the probability that you get a quarter? Note that there are 20 quarters and 20 dimes in all.

28. Suppose that the number of children in a family has the following distribution:

Number of children	0	1	2	3	4
Probability	0.15	0.25	0.3	0.2	0.1

Assume that each child is independently a girl or a boy with probability  $1/2$  each. If a family is picked at random what is the chance it has exactly two girls.

29. A student is taking a multiple-choice test in which each question has four possible answers. She knows the answers to 50% of the questions, can narrow the choices down to two 30% of the time, and does not know anything about 20% of the questions. What is the probability that she will correctly answer a question chosen at random from the test?

30. A student is taking a multiple-choice test in which each question has four possible answers. She knows the answers to 5 of the questions, can narrow the choices down to 2 in 3 cases, and does not know anything about 2 of the questions. What is the probability that she will correctly answer (a) 10, (b) 9, (c) 8, (d) 7, (e) 6, and (f) 5 questions?

31. Two boys, Charlie and Doug, take turns rolling two dice with Charlie going first. If Charlie rolls a 6 before Doug rolls a 7 he wins. What is the probability that Charlie wins?

32. Three boys take turns shooting a basketball and have probabilities 0.2, 0.3, and 0.5 of scoring a basket. Compute the probabilities for each boy to get the first basket.

33. Change the second and third probabilities in the last problem so that each boy has an equal chance of winning.

**Bayes' formula**

34. 5% of men and 0.25% of women are color blind. Assuming that there are an equal number of men and women, what is the probability that a color-blind person is a man?
35. The alpha fetal protein test is meant to detect spina bifida in unborn babies, a condition that affects 1 out of 1,000 children who are born. The literature on the test indicates that 5% of the time a healthy baby will cause a positive reaction. We will assume that the test is positive 100% of the time when spina bifida is present. Your doctor has just told you that your alpha fetal protein test was positive. What is the probability that your baby has spina bifida?
36. Binary digits, that is, 0's and 1's, are sent down a noisy communications channel. They are received as sent with probability 0.9 but errors occur with probability 0.1. Assuming that 0's and 1's are equally likely, what is the probability that a 1 was sent given that we received a 1?
37. To improve the reliability of the channel described in the last example, we repeat each digit in the message three times. What is the probability that 111 was sent given that (a) we received 101? (b) We received 000?
38. Two hunters shoot at a deer, which is hit by exactly one bullet. If the first hunter hits his target with probability 0.3 and the second with probability 0.6, what is the probability that the second hunter killed the deer? The answer is not  $2/3$ . Do you think the answer is larger or smaller?
39. A cab was involved in a hit-and-run accident at night. Two cab companies green and blue operate 85% and 15% of the cabs in the city respectively. A witness identified the cab as blue. However, in a test only 80% of witnesses were able to correctly identify the cab color. Given this what is the probability that the cab involved in the accident was blue?
40. A student goes to class on a snowy day with probability 0.4, but on a nonsnowy day attends with probability 0.7. Suppose that 20% of the days in February are snowy. What is the probability that it snowed on February 7 given that the student was in class on that day?
41. A company gave a test to 100 salesman, 80 with good sales records and 20 with poor sales records. 60% of the good salesman passed the test but only 30% of the poor salesmen did. Andy passed the test. Given this, what is the probability that he is a good salesman?
42. A company rates 80% of its employees as satisfactory and 20% as unsatisfactory. Personnel records indicate that 70% of the satisfactory workers had

prior experience but only 40% of the unsatisfactory workers did. If a person with previous work experience is hired, what is the probability that they will be a satisfactory worker?

43. A golfer hits his drive in the fairway with probability 0.7. When he hits his drive in the fairway he makes par 80% of the time. When he doesn't, he makes par only 30% of the time. He just made par on a hole. What is the probability that he hit his drive in the fairway?

44. You are about to have an interview for Harvard Law School. 60% of the interviewers are conservative and 40% are liberal. 50% of the conservatives smoke cigars but only 25% of the liberals do. Your interviewer lights up a cigar. What is the probability that he is a liberal?

45. Five pennies are sitting on a table. One is a trick coin that has heads on both sides, but the other four are normal. You pick up a penny at random and flip it four times, getting heads each time. Given this, what is the probability that you picked up the two-headed penny?

46. One slot machine pays off  $1/2$  of the time, while another pays off  $1/4$  of the time. We pick one of the machines and play it 6 times, winning 3 times. What is the probability we are playing the machine that pays off only  $1/4$  of the time?

47. A student is taking a multiple-choice exam in which each question has four possible answers. She knows the answers to 60% of the questions and guesses at the others. What is the probability that she guessed given that she got question 12 right?

48. 20% of people are "accident-prone" and have a probability 0.15 of having an accident in a 1-year period in contrast to a probability of 0.05 for the other 80% of people. (a) If we pick a person at random, what is the probability that they will have an accident this year? (b) What is the probability a person is accident-prone if they had an accident last year? (c) What is the probability that they will have an accident this year if they had one last year?

49. One die has 4 red and 2 white sides; a second has 2 red and 4 white sides. (a) If we pick a die at random and roll it, what is the probability that the result is a red side? (b) If the first result is a red side and we roll the same die again, what is the probability of a second red side?

50. A particular football team is known to run 40% of its plays to the left and 60% to the right. When the play goes to the right, the right tackle shifts his stance 80% of the time, but does so only 10% of the time when the play goes to the left. As the team sets up for the play the right tackle shifts his stance. What is the probability that the play will go to the right?

51. A professor gives a test to 90 calculus students, 60 who have had calculus before and 30 who have not. 30% of the students in the first group not an *A*, but only 10% in the second group did. Barbara got an *A*. What is the probability she had calculus before?
52. You are a serious student who studies on Friday nights but your roommate goes out and has a good time. 40% of the time he goes out with his girlfriend; 60% of the time he goes to a bar. 30% of the times when he goes out with his girlfriend he spends the night at her apartment. 40% of the times when he goes to a bar he gets in a fight and gets thrown in jail. You wake up on Saturday morning and your roommate is not home. What is the probability that he is in jail?
53. Two masked robbers try to rob a crowded bank during the lunch hour but the teller presses a button that sets off an alarm and locks the front door. The robbers, realizing they are trapped, throw away their masks and disappear into the chaotic crowd. Confronted with 40 people claiming they are innocent, the police give everyone a lie detector test. Suppose that guilty people are detected with probability 0.95, and innocent people appear to be guilty with probability 0.01. What is the probability that Mr. Jones is guilty given that the lie detector says he is?
54. Three bags lie on the table. One has two gold coins, one has two silver coins, and one has one silver and one gold. You pick a bag at random, and pick out one coin. If this coin is gold, what is the probability that you picked from the bag with two gold coins?
55. In a certain city, 30% of the people are Conservatives, 50% are Liberals, and 20% are Independents. In a given election,  $\frac{2}{3}$  of the Conservatives voted, 80% of the Liberals voted, and 50% of the Independents voted. If we pick a voter at random what is the probability that he or she is Liberal?
56. An undergraduate student has asked a professor for a letter of recommendation. He estimates that the probability he will get the job is 0.8 with a strong letter, 0.4 with a medium letter, and 0.1 with a weak letter. He also believes that the probabilities that the letter will be strong, medium, or weak are 0.5, 0.3, and 0.2. What is the probability that the letter was strong given that he got the job.
57. A group of 20 people go out to dinner. 10 go to an Italian restaurant, 6 to a Japanese restaurant, and 4 to a French restaurant. The fractions of people satisfied with their meals were  $\frac{4}{5}$ ,  $\frac{2}{3}$ , and  $\frac{1}{2}$  respectively. The next day the person you are talking to was satisfied with what they ate. What is the probability that they went to the Italian restaurant? The Japanese restaurant? The French restaurant?

## Conditional Probability

58. 1 out of 1,000 births results in fraternal twins; 1 out of 1,500 births results in identical twins. Identical twins must be the same sex but the sexes of fraternal twins are independent. If two girls are twins, what is the probability that they are fraternal twins?

59. Consider the following data on traffic accidents:

Age group	% of drivers	Accident probability
16-25	15	0.10
26-45	35	0.04
46-65	35	0.06
>65	15	0.08

Calculate (a) the probability that a randomly chosen driver will have an accident this year, and (b) the probability that a driver is between 46 and 65 given that they had an accident.

**Joint distributions**

60. Suppose we draw two tickets from a hat that contains tickets numbered 1, 2, 3, 4. Let  $X$  be the first number drawn and  $Y$  be the second. Find the joint distribution of  $X$  and  $Y$ .

61. Suppose we roll one die repeatedly and let  $N_i$  be the number of the roll on which  $i$  first appears. Find the joint distribution of  $N_1$  and  $N_6$ .

62. Compute (a)  $P(X = 1|Y = 1)$  and (b)  $P(X = 2|Y = 2)$  for the following joint distribution:

$Y$	$X = 1$	2	3
1	0.1	0.2	0.3
2	0.15	0.15	0
3	0.05	0	0.05

63. Compute (a)  $P(X = 2|Y = 3)$  and (b)  $P(Y = 3|X = 3)$  for the following joint distribution:

$Y$	$X = 1$	2	3
1	0.2	0.15	0.05
2	0.10	0	0.10
3	0.05	0.15	0.20

64. Using the clues given below, fill in the rest of the joint distribution. There is only one answer.

$Y$	$X = 0$	3	6
1	?	?	?
2	0.1	0.05	?

(a)  $P(Y = 2|X = 0) = 1/4$ , (b)  $X$  and  $Y$  are independent.



## 3.5 Exercises

65. Using the clues given below, fill in the rest of the joint distribution. There is only one answer:

Y	X = 1	2	3
1	?	?	?
2	?	0	?
3	0	?	0

For  $k = 1, 2, 3$ , (a)  $P(Y = 1|X = k) = 2/3$ , (b)  $P(X = k|Y = 1) = k/6$ .

66. Fill in the rest of the joint distribution so that  $X$  and  $Y$  are independent. There are two possible answers:

Y	X = 0	1
0	?	2/9
1	2/9	?