

## 1.7 Exercises

### Basic definitions

1. A man receives presents from his three children, Allison, Betty, and Chelsea. To avoid disputes he opens the presents in a random order. What are the possible outcomes?
2. Suppose we pick a number at random from the phone book and look at the last digit. (a) What is the set of outcomes and what probability should be assigned to each outcome? (b) Would this model be appropriate if we were looking at the first digit?
3. Two students arrive late for a math final exam with the excuse that their car had a flat tire. Suspicious, the professor says "each one of you write down on a piece of paper which tire was flat." What is the probability that both students pick the same tire?
4. Suppose we roll a red die and a green die. What is the probability that the number on the red die is larger than the number on the green die?
5. Two dice are rolled. What is the probability that (a) the two numbers will differ by 1 or less and (b) the maximum of the two numbers will be 5 or larger?
6. If we flip a coin 5 times, what is the probability that the number of heads is an even number (that is, divisible by 2)?
7. The 1987 World Series was tied at two games a piece before the St. Louis Cardinals won the fifth game. According to the Associated Press, "The numbers of history support the Cardinals and the momentum they carry. Whenever the series has been tied 2-2 the team that won the fifth game won the series 71% of the time." If momentum is not a factor and each team has a 50% chance of winning each game, what is the probability that the Game 5 winner will win the series?
8. Two boys are repeatedly playing a game that they each have probability  $1/2$  of winning. The first person to win 5 games wins the match. What is the probability that Al will win if (a) he has won 4 games and Bobby has won 3 and (b) he leads by a score of 3 games to 2?
9. 20 families live in a neighborhood: 4 have 1 child, 8 have 2 children, 5 have 3 children, and 3 have 4 children. If we pick a child at random, what is the probability that they come from a family with 1, 2, 3, 4 children?
10. In Galileo's time people thought that when three dice were rolled, a sum of 9 and a sum of 10 had the same probability since each could be obtained

in 6 ways:

$$9: 1 + 2 + 6, 1 + 3 + 5, 1 + 4 + 4, 2 + 2 + 5, 2 + 3 + 4, 3 + 3 + 3$$

$$10: 1 + 3 + 6, 1 + 4 + 5, 2 + 4 + 4, 2 + 3 + 5, 2 + 2 + 6, 3 + 3 + 4$$

Compute the probabilities of these sums and show that 10 is a more likely total than 9.

11. Suppose we roll three dice. Compute the probability that the sum is (a) 3, (b) 4, (c) 5, (d) 6, (e) 7, (f) 8, (g) 9, (h) 10.

12. In a group of students, 25% smoke cigarettes, 60% drink alcohol, and 15% do both. What fraction of students have at least one of these bad habits?

13. In a group of 320 high school graduates, only 160 went to college but 100 of the 170 men did. How many women did not go to college?

14. In the freshman class, 62% of the students take math, 49% take science, and 38% take both science and math. What percentage takes at least one science or math course?

15. 24% of people have American Express cards, 61% have Visa cards, and 8% have both. What percentage of people have at least one credit card?

16. Suppose  $\Omega = \{a, b, c\}$ ,  $P(\{a, b\}) = 0.7$ , and  $P(\{b, c\}) = 0.6$ . Compute the probabilities of  $\{a\}$ ,  $\{b\}$ , and  $\{c\}$ .

17. Suppose  $A$  and  $B$  are disjoint with  $P(A) = 0.3$  and  $P(B) = 0.5$ . What is  $P(A^c \cap B^c)$ ?

18. Given two events  $A$  and  $B$  with  $P(A) = 0.4$  and  $P(B) = 0.7$ , what are the maximum and minimum possible values for  $P(A \cap B)$ ?

### Independence

19. Suppose we draw 2 cards out of a deck of 52. Let  $A =$  "the first card is an ace" and  $B =$  "the second card is a spade." Are  $A$  and  $B$  independent?

20. A family has 3 children, each of whom is a boy or a girl with probability  $1/2$ . Let  $A =$  "there is at most 1 girl"  $B =$  "the family has children of both sexes." (a) Are  $A$  and  $B$  independent? (b) Are  $A$  and  $B$  independent if the family has 4 children?

21. Suppose we roll a red and a green die. Let  $A =$  "the red die shows a 2 or a 5" and  $B =$  "the sum of the two dice is at least 7." Are  $A$  and  $B$  independent?



22. Roll two dice. Let  $A$  = "the sum is even" and  $B$  = "the sum is divisible by 3," that is,  $B = \{3, 6, 9, 12\}$ . Are  $A$  and  $B$  independent?
23. Roll two dice. Let  $A$  = "the first die is odd,"  $B$  = "the second die is odd," and  $C$  = "the sum is odd." Show that these events are pairwise independent but not independent.
24. Nine children are seated at random in three rows of three desks. Let  $A$  = "Al and Bobby sit in the same row" and  $B$  = "Al and Bobby both sit at one of the four corner desks." Are  $A$  and  $B$  independent?
25. Two students, Alice and Betty, are registered for a statistics class. Alice attends 80% of the time, Betty 60% of the time, and their absences are independent. On a given day, what is the probability that (a) at least one of these students is in class and (b) exactly one of them is there?
26. Let  $A$  and  $B$  be two independent events with  $P(A) = 0.4$  and  $P(A \cup B) = 0.64$ . What is  $P(B)$ ?
27. Three students each have probability  $1/3$  of solving a problem. What is the probability that at least one of them will solve the problem?
28. Three independent events have probabilities  $1/4$ ,  $1/3$ , and  $1/2$ . What is the probability that exactly one will occur?
29. Three missiles are fired at a target. They will hit it with probabilities 0.2, 0.4, and 0.6. Find the probability that the target is hit by (a) three, (b) two, (c) one, and (d) no missiles.
30. Three couples that were invited to dinner will independently show up with probabilities 0.9, 0.89, and 0.75. Let  $N$  be the number of couples that show up. Calculate the probability  $P(N)$  with  $N = 3, 2, 1, 0$ .
31. A college student takes 4 courses a semester for 8 semesters. In each course she has a probability  $1/2$  of getting an A. Assuming her grades in different courses are independent, what is the probability that she will have at least one semester with all A's?
32. When Al and Bob play tennis, Al wins a set with probability 0.7 while Bob wins with probability 0.3. What is the probability that Al will be the first to win (a) two sets and (b) three sets?
33. Chevalier de Méré made money betting that he could "roll at least one 6 in 4 tries." When people got tired of this wager he changed it to "roll at least one double 6 in 24 tries," but then he started losing money. Compute the probabilities of winning these two bets.

34. Samuel Pepys wrote to Isaac Newton: "What is more likely, (a) at least one 6 in 6 rolls of one die or (b) at least two 6's in 12 rolls?" Compute the probabilities of these events.

### Distributions

35. Suppose we roll two dice and let  $X$  and  $Y$  be the two numbers that appear. Find the distribution of  $|X - Y|$ .

36. Suppose we roll three tetrahedral dice that have 1, 2, 3, and 4 on their four sides. Find the distribution for the sum of the three numbers.

37. We roll two six-sided dice, one with sides 1, 2, 2, 3, 3, 4 and the other with sides 1, 3, 4, 5, 6, 8. What is the distribution of the sum?

38. How many children should a family plan to have so that the probability of having at least one child of each sex is  $\geq 0.95$ ?

39. How many times should a coin be tossed so that the probability of at least one head is  $\geq 99\%$ ?

### Expected value

40. You want to invent a gambling game in which a person rolls two dice and is paid some money if the sum is 7, but otherwise he loses his money. How much should you pay him for winning a \$1 bet if you want this to be a fair game, that is, to have expected value 0?

41. A bet is said to carry 3 to 1 odds if you win \$3 for each \$1 you bet. What must the probability of winning be for this to be a fair bet?

42. A lottery has one \$100 prize, two \$25 prizes, and five \$10 prizes. What should you be willing to pay for a ticket if 100 tickets are sold?

43. In a popular gambling game, three dice are rolled. For a \$1 bet you win \$1 for each 6 that appears (plus your dollar back). If no 6 appears you lose your dollar. What is your expected value?

44. A roulette wheel has slots numbered 1 to 36 and two labeled with 0 and 00. Suppose that all 38 outcomes have equal probabilities. Compute the expected values of the following bets. In each case you bet one dollar and when you win you get your dollar back in addition to your winnings. (a) You win \$1 if one of the numbers 1 through 18 comes up. (b) You win \$2 if the number that comes up is divisible by 3 (0 and 00 do not count). (c) You win \$35 if the number 7 comes up.



45. In the Las Vegas game Wheel of Fortune, there are 54 possible outcomes. One is labeled "Joker," one "Flag," two "20," four "10," seven "5," fifteen "2," and twenty-four "1." If you bet \$1 on a number you win that amount of money if the number comes up (plus your dollar back). If you bet \$1 on Flag or Joker, you win \$40 if that symbol comes up (plus your dollar back). What bets have the best and worst expected value here?

46. Sic Bo is an ancient Chinese dice game played with three dice. One of the possibilities for betting in the game is to bet "big." For this bet, you win if the total  $X$  is 11, 12, 13, 14, 15, 16, or 17, except when there are three 4's or three 5's. On a \$1 bet on big, you win \$1 plus your dollar back if it happens. What is your expected value?

47. Five people play a game of "odd man out" to determine who will pay for the pizza they ordered. Each flips a coin. If only one person gets heads (or tails) while the other four get tails (or heads) then he is the odd man and has to pay. Otherwise they flip again. What is the expected number of tosses needed to determine who will pay?

48. A man and his wife decide that they will keep having children until they have one of each sex. Ignoring the possibility of twins and supposing that each trial is independent and results in a boy or a girl with probability  $1/2$ , what is the expected value of the number of children they will have?

49. An unreliable clothes dryer dries your clothes and takes 20 minutes with probability 0.6 and buzzes for 4 minutes and does nothing with probability 0.4. If we assume that successive trials are independent and that we patiently keep putting our money in to try to get it to work, what is the expected time we need to get our clothes dry?

### Moments and variance

50. A random variable has  $P(X = x) = x/15$  for  $x = 1, 2, 3, 4, 5$ , and 0 otherwise. Find the mean and variance of  $X$ .

51. Find the mean and variance of the number of games in the World Series. Recall that it is won by the first team to win four games and assume that the outcomes are determined by flipping a coin.

52. Suppose we pick a month at random from a non-leap year calendar and let  $X$  be the number of days in the month. Find the mean and variance of  $X$ .

53. The Elm Tree golf course in Cortland, NY, is a par 70 layout with 3 par fives, 5 par threes, and 10 par fours. Find the mean and variance of par on this course.

54. In a group of five items, two are defective. Find the distribution of  $N$ , the number of draws we need to find the first defective item. Find the mean and variance of  $N$ .

55. Can we have a random variable with  $EX = 3$  and  $EX^2 = 8$ ?

56. Suppose  $P(X \in \{1, 2, 3\}) = 1$  and  $EX = 2.5$ . What are the smallest and largest possible values for the variance?