Chap 6 #30

$T \in \mathcal{L}(V,W)$. (a) T injective $\Leftrightarrow$ T^* surjective
(b) T surjective $\Leftrightarrow$ T^* injective.

Pf: (a) T injective $\Leftrightarrow$ $\mathrm{null}(T) = \{0\}$

$$\overset{\shortparallel}{\Leftrightarrow} \left(\mathrm{range}(T^*)\right)^{\perp} = \{0\}$$

$$\Leftrightarrow \mathrm{range}(T^*) = \{0\}^{\perp} = V$$

$$\Leftrightarrow T^* \text{ surjective}$$

(b) By (a), and since $(T^*)^* = T$, we have:

$$T^* \text{ injective} \Leftrightarrow (T^*)^* = T \text{ surjective}.$$

Chap 7 #1a

$\mathcal{P}_2(\mathbb{R})$ with IP $\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx$

Define: $T(a_0 + a_1 x + a_2 x^2) = a_1 x$

Show: T not self-adjoint.

If T were self-adjoint, then there must hold:

$$(*) \quad \langle Tp, q \rangle = \langle p, Tq \rangle, \quad \forall p, q \in \mathcal{P}_2(\mathbb{R})$$

If we find $p, q \in \mathcal{P}_2(\mathbb{R})$ such that $(*)$ fails, we would have a contradiction, and T would not be self-adjoint.

Let $p(x) = 1$, $q(x) = x$. Then $\left.\begin{array}{l}\langle Tp, q \rangle = \int_0^1 0 \cdot x\,dx = 0 \\ \langle p, Tq \rangle = \int_0^1 1 \cdot 1\,dx = 1\end{array}\right\} \neq$.

(b) (Not graded!)

$$M(T, \{1, x, x^2\}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{transpose of the complex conjugate of } M(T, \{1, x, x^2\}).$$

But still we know from (a) that T is not self-adjoint.

This appears to contradict Proposition 6.47 which says that $M(T^*, \{1, x, x^2\})$ also equals the above matrix.

The key point here is that Prop 6.47 requires the basis to be orthonormal, which $\{1, x, x^2\}$ is not.

You know this from a previous HW problem (chap 6, #10) where you actually applied Gauss-Schmidt to $\{1, x, x^2\}$ in order to construct an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

You could calculate $M(T, \text{orthonormal basis})$ and will find that this matrix does <u>not</u> equal its complex conjugate transpose.