Chap 6

#10. On $P_2(\mathbb{R})$, the IP: $\langle p,q\rangle = \int_0^1 p(x)q(x)\,dx$

Apply Gram-Schmidt to the basis $(1, x, x^2)$ to produce an orthonormal basis of $P_2(\mathbb{R})$.

Sol: $v_1 = 1$, $v_2 = x$, $v_3 = x^2$

- $e_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{1}{\left(\int_0^1 1\cdot 1\,dx\right)^{1/2}} = \dfrac{1}{1} = 1$

- $e_2 = \dfrac{v_2 - \langle v_2, e_1\rangle e_1}{\|v_2 - \langle v_2, e_1\rangle e_1\|}$, $\langle v_2, e_1\rangle = \int_0^1 x\cdot 1\,dx = \frac{1}{2}$

$\Rightarrow v_2 - \langle v_2, e_1\rangle e_1 = x - \frac{1}{2}\cdot 1 = x - \frac{1}{2}$

$\|v_2 - \langle v_2, e_1\rangle e_1\| = \left(\int_0^1 (x-\frac{1}{2})^2\,dx\right)^{1/2} = \left(\frac{1}{3}(x-\frac{1}{2})^3\Big|_0^1\right)^{1/2}$

$= \left(\frac{1}{3}\cdot\frac{1}{8} - \frac{1}{3}\cdot(-\frac{1}{8})\right)^{1/2} = \left(\frac{1}{12}\right)^{1/2} = \frac{1}{2\sqrt{3}}$

$\Rightarrow e_2 = \sqrt{3}(2x-1)$

- $e_3 = \dfrac{v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2}{\|\ \text{—}\ \|}$, $\langle v_3, e_1\rangle = \int_0^1 x^2\cdot 1\,dx = \frac{1}{3}$

$\langle v_3, e_2\rangle = \int_0^1 x^2\cdot\sqrt{3}(2x-1)\,dx = \sqrt{3}\left(2\cdot\frac{1}{4} - \frac{1}{3}\right) = \frac{\sqrt{3}}{6}$

$\Rightarrow v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2 = x^2 - \frac{1}{3}\cdot 1 - \frac{\sqrt{3}}{6}\cdot\sqrt{3}(2x-1)$

$= x^2 - x + \frac{1}{6}$

$\|v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2\| = \left(\int_0^1 (x^2 - x + \frac{1}{6})^2\,dx\right)^{1/2}$

$= \left(\int_0^1 x^4 + x^2 + \frac{1}{36} - 2x^3 - \frac{x}{3} + \frac{x^2}{3}\,dx\right)^{1/2}$

$= \left(\frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{2}{4} - \frac{1}{6} + \frac{1}{9}\right)^{1/2} = \left(\frac{1}{5} + \frac{12+1-18-6+4}{36}\right)^{1/2}$

$= \left(\frac{1}{5} + \frac{-7}{36}\right)^{1/2} = \left(\frac{36-35}{180}\right)^{1/2} = \frac{1}{\sqrt{180}} = \frac{1}{6\sqrt{5}}$

$\Rightarrow e_3 = \sqrt{5}(6x^2 - 6x + 1)$

Thus, $\{e_1, e_2, e_3\} = \{1, \sqrt{3}(2x-1), \sqrt{5}(6x^2 - 6x+1)\}$ is orthonormal basis of $P_2(\mathbb{R})$.

#11 What happens when Gram-Schmidt is applied to a linearly dependent set.

It fails! Suppose that $\{v_1, v_2, \ldots, v_m\}$ is linearly dependent.

- If $v_1 = 0$, then e_1 cannot be defined as $\frac{v_1}{\|v_1\|}$.

- If $v_1 \neq 0$, but $\{v_1 \ldots v_m\}$ is linearly dependent, then the procedure outlined in the proof of the Gram-Schmidt process works for all j such that $\{v_1, v_2, \ldots, v_j\}$ is linearly <u>in</u>dependent.

However, let j^* be the smallest integer such that $\{v_1 \ldots v_{j^*}\}$ is linearly dependent. Then $v_{j^*} \in \text{span}\{v_1, \ldots, v_{j^*-1}\}$
$= \text{span}\{e_1, \ldots, e_{j^*-1}\}$.

Try to construct $e_{j^*} = \dfrac{v_{j^*} - (\langle v_{j^*}, e_1\rangle e_1 + \cdots + \langle v_{j^*}, e_{j^*-1}\rangle e_{j^*-1})}{\| \qquad\qquad \|}$ $(*)$

But since $v_{j^*} \in \text{span}\{e_1, \ldots, e_{j^*-1}\}$ and $\{e_1, \ldots, e_{j^*-1}\}$ is an orthonormal list, it follows that

$$v_{j^*} = \langle v_{j^*}, e_1\rangle e_1 + \langle v_{j^*}, e_2\rangle e_2 + \cdots + \langle v_{j^*-1}, e_{j^*-1}\rangle e_{j^*-1}$$

by Thm 6.17

But then the denominator in $(*)$ is 0, hence e_{j^*} cannot be defined.

#21 In $\mathbb{R}^4$ let $U = \operatorname{span}\big((1,1,0,0),(1,1,1,2)\big)$

Find $u \in U$ such that $\|u-(1,2,3,4)\|$ is as small as possible.

Sol: Note first that $(1,2,3,4) \notin U$. Indeed, any vector in U is such that its first 2 entries are the same, which is not the case for $(1,2,3,4)$.

We know that the sought-after $u \in U$ is the orthogonal projection onto U, of the vector $(1,2,3,4)$.

$$\Rightarrow u = P_U(1,2,3,4)$$

$$= \langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2$$

Here $v = (1,2,3,4)$ and $\{e_1, e_2\}$ is an orthonormal basis of U. To get such a basis we apply Gram-Schmidt.

$$e_1 = \frac{(1,1,0,0)}{\|(1,1,0,0)\|} = \frac{1}{\sqrt{2}}(1,1,0,0)$$

$$e_2 = \frac{(1,1,1,2) - \langle (1,1,1,2), \frac{1}{\sqrt2}(1,1,0,0)\rangle \cdot \frac{1}{\sqrt2}(1,1,0,0)}{\| \quad \text{———} \quad \|}$$

$$= \frac{(1,1,1,2) - \frac{2}{\sqrt2}\cdot\frac{1}{\sqrt2}(1,1,0,0)}{\| \text{——} \|} = \frac{(0,0,1,2)}{\|(0,0,1,2)\|} = \frac{1}{\sqrt5}(0,0,1,2)$$

$$\Rightarrow u = \left\langle (1,2,3,4), \frac{1}{\sqrt2}(1,1,0,0)\right\rangle \cdot \frac{1}{\sqrt2}(1,1,0,0)$$
$$+ \left\langle (1,2,3,4), \frac{1}{\sqrt5}(0,0,1,2)\right\rangle \cdot \frac{1}{\sqrt5}(0,0,1,2)$$
$$= \frac{3}{2}(1,1,0,0) + \frac{11}{5}(0,0,1,2) = \frac{1}{10}(15,15,22,44)$$