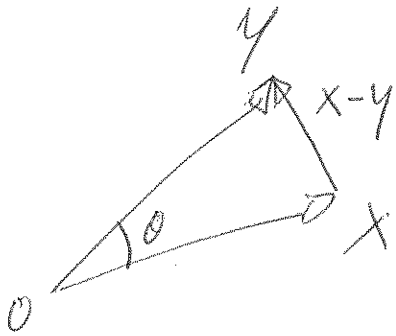
HW VII : Chap 6, #1, #3

#1 If x and y are nonzero vectors in $\mathbb{R}^2$, then prove:

$$\langle x, y\rangle = \|x\|\,\|y\| \cos\theta$$

[Here $\langle\cdot,\cdot\rangle$ is the Euclidean I.P. on $\mathbb{R}^2$]

where θ is the angle between x and y.



Law of Cosines:

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\|\cos\theta$$

$$\langle x-y, x-y\rangle = \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2$$
$$= \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$$

because $\langle y,x\rangle = \langle x,y\rangle$

(no complex conjugation since $\mathbb{R}^2$ is a <u>real</u> vector space)

Thus: $\cancel{\|x\|^2} - 2\langle x,y\rangle + \cancel{\|y\|^2} = \cancel{\|x\|^2} + \cancel{\|y\|^2} - 2\|x\|\,\|y\|\cos\theta$

$$\Longrightarrow \langle x,y\rangle = \|x\|\,\|y\|\cos\theta.$$

#3 Prove: $\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} j a_j^2\right)\left(\sum_{j=1}^{n} \frac{b_j^2}{j}\right)$ 2

for any real numbers $a_j, b_j, j=1\ldots n$.

Pf: Apply Cauchy-Schwarz to the following 2 vectors in $\mathbb{R}^n$, using the Euclidean IP on $\mathbb{R}^n$:

$$u = (\sqrt{1}\,a_1, \sqrt{2}\,a_2, \ldots, \sqrt{n}\,a_n)$$

$$v = \left(\frac{b_1}{\sqrt{1}}, \frac{b_2}{\sqrt{2}}, \ldots, \frac{b_n}{\sqrt{n}}\right)$$

$\Rightarrow |\langle u,v\rangle| \leq \|u\|\,\|v\|$ and thus $|\langle u,v\rangle|^2 \leq \|u\|^2\,\|v\|^2$

But $\langle u,v\rangle = \sum_{j=1}^{n} (\sqrt{j}\,a_j)\cdot\left(\frac{b_j}{\sqrt{j}}\right) = \sum_{j=1}^{n} a_j b_j$

$$\|u\|^2 = (\sqrt{1}\,a_1)^2 + (\sqrt{2}\,a_2)^2 + \cdots + (\sqrt{n}\,a_n)^2 = \sum_{j=1}^{n} j a_j^2$$

Similarly, $\|v\|^2 = \sum_{j=1}^{n} \frac{b_j^2}{j}$

Therefore: $\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} j a_j^2\right)\left(\sum_{j=1}^{n} \frac{b_j^2}{j}\right)$.