#15 $F = \mathbb{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{C})$, $a \in \mathbb{C}$.

Then: a is eigenvalue of $p(T) \Longleftrightarrow$ there is an eigenvalue λ of T such that $a = p(\lambda)$

Pf: ($\Longleftarrow$) Let $Tv \overset{(*)}{=} \lambda v$ for some $v \neq 0$ such that $a = p(\lambda)$.
We need to show that a is an eigenvalue of $p(T)$.
First, what is $p(T)$? If $p \in \mathcal{P}(\mathbb{C})$, then $p(z) = a_m z^m + \dots + a_1 z + a_0$ for certain scalars $a_0 \dots a_m$
and then $p(T)$ is the linear operator $a_m T^m + \dots + a_1 T + a_0 I$.

Note that this implies that $T^m v = \lambda^m v$, for all $m = 1, 2, \dots$

We need to find a nonzero vector w in V such that $p(T)w = aw$
Claim: $w = v$ will work.

$$\text{Indeed } [p(T)](v) = (a_m T^m + \dots + a_1 T + a_0 I)(v) = a_m T(v)^m + \dots + a_1 T(v) + a_0 v$$
$$\overset{(*)}{=} a_m \lambda^m v + \dots + a_1 \lambda v + a_0 v$$
$$= p(\lambda) v$$
$$= av$$

($\Longrightarrow$) Let a be an eigenvalue of $p(T)$:
Then $(p(T))(v) = av$ for some $v \neq 0$. We assume that $p(z) = a_m z^m + \dots + a_1 z + a_0$ for some $m \geq 0$ (0 is allowed)

$$\text{Thus } a_m T^m(v) + a_{m-1} T^{m-1}(v) + \dots + a_1 T(v) + (a_0 - a)v = 0 \quad (**)$$

Define $q \in \mathcal{P}(z)$ to be the polynomial $q(z) = a_m z^m + \dots + a_1 z + (a_0 - a)$

Consider 2 cases:

1. If $m > 0$, by the fundamental theorem of algebra, we can factor $q(z)$:
$$q(z) = a_m (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_m)$$
Thus we have that $a_m (T - \lambda_1 I)(T - \lambda_2 I) \dots (T - \lambda_m I)(v) = 0$ by $(**)$
Since $v \neq 0$ this implies that at least one of the $T - \lambda_i I$ is not injective
$\Rightarrow \exists\, w \neq 0$ such that $(T - \lambda_i I)(w) = 0 \Rightarrow Tw = \lambda_i w$
But then $(q(T))(w) = 0 \Rightarrow a_m T^m w + \dots + a_1 Tw + a_0 w = aw$
$\Rightarrow (a_m \lambda_i^m + \dots + a_1 \lambda_i + a_0) w = aw$ or $p(\lambda_i) w = aw$
and since $w \neq 0$, $p(\lambda_i) = a$ must hold.
2. If $m = 0$, then $(a_0 - a)v = 0$ with $v \neq 0 \Rightarrow a_0 = a$, But then $p(z) = a_0 = a$, for any $z \in \mathbb{C}$,
and therefore certainly for an eigenvalue λ of T (which exists by Thm 5.13)

#16 : $\begin{cases} V = \mathbb{R}^2 \\ T(x,y) = (-y, x). \\ p(z) = z^2, \end{cases}$

Consider $p(T) = T^2$

So $\big(p(T)\big)(x,y) = T^2(x,y)$

$$= T(-y,x)$$
$$= (-x,-y) = (-1)(x,y)$$

Thus, $a=-1$ is (the only) eigenvalue of $p(T)$

However, it is clear that $a=-1$, cannot equal $p(\lambda)=\lambda^2$ for any real number λ.

In fact, according to #15, λ should be an eigenvalue of T, but we know from the discussion in class that T has no (real) eigenvalues.

(See also p. 78 for this discussion)

Remark: T amounts to a rotation of $90°$ counterclockwise

Geometric interpretation

T^2 ——— of $180°$ ———

We know that T cannot have (real) eigenvalues because no line through the origin is invariant under T.

But every line through the origin is obviously invariant under T^2 since every point (x,y) is mapped to its reflection with respect to the origin, $(-x,-y)$, which is still on the line through (x,y) and the origin.

#21 $P \in \mathcal{L}(V)$ with $P^2 = P$.

Prove : $V = \text{null}(P) \oplus \text{range}(P)$

Pf: • Let's first prove that $V = \text{null}(P) + \text{range}(P)$

Pick $v \in V$. Then $v = \underbrace{(v - Pv)}_{\in \text{null}(P)} + \underbrace{Pv}_{\in \text{range}(P)}$ (This is obvious)

indeed: $P(v - Pv) = Pv - P^2v \underset{P^2=P}{=} Pv - Pv = 0$

Thus, $V = \text{null}(P) + \text{range}(P)$

• Let's now prove that the sum is direct:
$V = \text{null}(P) \oplus \text{range}(P)$

It suffices to prove that $\text{null}(P) \cap \text{range}(P) = \{0\}$

So let $w \in \text{null}(P) \cap \text{range}(P)$.

$\Rightarrow Pw = 0$ and $w = Pv$, for some $v \in V$

$\Rightarrow 0 = Pw = P(Pv) = P^2 v \underset{P=P^2}{=} Pv$

Thus $Pv = 0$. But then $w = Pv = 0$

$\Rightarrow w = 0$.