Chap 3

#23 V finite-dim V.S. and $S, T \in \mathcal{L}(V)$.

Then $ST = I \iff TS = I$

Pf: By symmetry it suffices to prove just 1 implication, say $\Rightarrow$.

Thus, let $ST = I$.

Then, since I is invertible, it is injective and surjective.

Thus, ST is injective and surjective.

Then $\left(\begin{array}{l} T \text{ must be injective} \\ \quad\text{and} \\ S \text{ must be surjective} \end{array}\right)$ (for otherwise ST would not have these properties)

But $T \in \mathcal{L}(V)$, hence by equivalence of injectivity and surjectivity, it follows that:

$\left(\begin{array}{l} T \text{ must be surjective} \\ \quad\text{and} \\ S \text{ must be injective} \end{array}\right.$

Thus, both T and S are injective and surjective and therefore they are invertible.

$\Rightarrow \exists!\ T^{-1} \in \mathcal{L}(V)$ and $\exists!\ S^{-1} \in \mathcal{L}(V)$: $TT^{-1} = T^{-1}T = I$, $SS^{-1} = S^{-1}S = I$

But since $ST = I$ and S has a unique inverse S^{-1}, it must hold that $T = S^{-1}$. Similarly, $S = T^{-1}$.

$\Rightarrow TS = I = ST$.

Chap 5

#2: $T \in \mathcal{L}(V)$. Let U_α, $\alpha \in I$ be collection of subspaces of V all invariant under T.
Then $\bigcap_{\alpha \in I} U_\alpha$ is invariant under T.

Pf: Let $u \in \bigcap_{\alpha \in I} U_\alpha$. Then $u \in U_\alpha$, all $\alpha \in I$.

$$\underset{\text{each } U_\alpha \text{ is } T\text{-invariant}}{\Rightarrow} Tu \in U_\alpha, \text{ all } \alpha \in I$$

$$\Rightarrow Tu \in \bigcap_{\alpha \in I} U_\alpha, \text{ hence } \bigcap_{\alpha \in I} U_\alpha \text{ is } T\text{-invariant.}$$

#4: $S, T \in \mathcal{L}(V)$ and $ST = TS$

Prove that $\text{null}(T - \lambda I)$ is S-invariant, $\forall \lambda \in F$.

Pf: Fix $\lambda \in F$ and pick $v \in \text{null}(T - \lambda I)$. Thus, $(T - \lambda I)(v) = 0_V$

We need to show that $Sv \in \text{null}(T - \lambda I)$ as well.

Let's see: Is $(T - \lambda I)(Sv) = 0$?

Well, $(T - \lambda I)(Sv) = TS(v) - \lambda I S(v) \underset{TS = ST}{=} ST(v) - \lambda S(v) = S(T - \lambda I)(v) \underset{\text{since } (T-\lambda I)(v) = 0}{= 0}$

So, yes.

#8: Find all eigenvalues, eigenvectors of the backward shift operator $T \in \mathcal{L}(F^\infty)$ (recall $T(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots)$)

$$Tz = \lambda z \Rightarrow (z_2, z_3, \ldots) = (\lambda z_1, \lambda z_2, \lambda z_3, \ldots)$$

Thus $\begin{cases} z_2 = \lambda z_1 \\ z_3 = \lambda z_2 \\ \vdots \end{cases}$

- Note: If $z_1 = 0$, then $z_2 = z_3 = \ldots = 0$. Thus, no eigenvector can have a zero entry for z_1 since eigenvectors must be nonzero vectors.
- We can assume $z_1 = 1$ by scaling. Then $(1, \lambda, \lambda^2, \ldots)$ is eigenvector with corresponding eigenvalue λ.

Thus: Every $\lambda \in F$ is an eigenvalue with eigenvectors $\text{span}\{(1, \lambda, \lambda^2, \ldots)\}$