HW 4 : Chap 3

#12 V, W finite-dim.

There is a surjective linear map $T: V \to W$ if and only if $\dim W \leq \dim V$

Pf: if : Assume $m = \dim W \leq \dim V = n$

$\to$ Fix basis $B_V = (v_1 \cdots v_n)$ for V

and basis $B_W = (w_1 \cdots w_m)$ for W

Define linear map $T: V \to W$ as follows

$$\begin{cases} T(v_1) = w_1 \\ T(v_2) = w_2 \\ \vdots \\ T(v_m) = w_m \\ T(v_{m+1}) = \cdots = T(v_n) = 0 \end{cases}$$

(Recall that it is enough to define the images of the basis vectors of V, in order to know the image of every vector in V under T)

Claim: T is surjective. Indeed, let $w \in W$. Then there are scalars

$a_1 \ldots a_m \in F$ such that: $w = a_1 w_1 + \cdots + a_m w_m$

Is this w the image under T of some vector v in V?

Yes, of $v = a_1 v_1 + \cdots + a_m v_m$, because clearly:

$$Tv = \underbrace{T(a_1 v_1 + \cdots + a_m v_m)}_{=w} = a_1 Tv_1 + \cdots + a_m Tv_m = a_1 w_1 + \cdots + a_m w_m$$

only if: This was already proved, see Corollary 3.6, p 46.

#13: V, W finite-dimensional
$U \subseteq V$ subspace
$\exists\, T \in \mathcal{L}(V, W) : \text{null}(T) = U \iff \dim U \geq \dim V - \dim W$ $(*)$

Pf: $\Rightarrow$ By dimension Thm, $\dim V = \dim(\text{null}(T)) + \dim(\text{range}(T))$
$= \dim U + \dim(\text{range}(T))$

$\Rightarrow \dim(\text{range}(T)) = \dim V - \dim U$
and of course $\dim(\text{range}(T)) \leq \dim W$ $\Big\} \Rightarrow$
$\dim W \geq \dim V - \dim U$, ie $(*)$

$\Leftarrow$ We need to find a linear map $T: V \to W$, whose nullspace is the subspace U.

Pick a basis $B_U = (u_1, \dots, u_n)$ of U. Extend this to a basis for V: $B_V = (u_1, \dots, u_n, v_1, \dots, v_m)$.

We define T by saying what the image under T of each basis vector in B_V is:

$$T(u_1) = T(u_2) = \dots = T(u_n) = 0$$

Notice that by $(*)$, we have that $\dim W \geq \underbrace{\dim V}_{= m+n} - \underbrace{\dim U}_{= n}$

$$\Rightarrow \dim W \geq m$$

Thus any basis of W has at least m basis vectors.
Suppose that $B_W = (w_1, \dots, w_m, \dots, w_{m+p})$ is a basis of W where $p \geq 0$ is an integer.

Define $T(v_1) = w_1$
$T(v_2) = w_2$
$\vdots$
$T(v_m) = w_m$

Thus we have now defined T on a basis of V, and therefore T is defined on all of V.

Last thing to check is that $\text{null}(T) = U$.

- That $U \subseteq \text{null}(T)$ is clear, because by definition every basis vector in B_U is mapped to 0 by T, and therefore every vector in U is also mapped to 0 by T.
- To show that $\text{null}(T) \subseteq U$, pick $z \in \text{null}(T)$

$\Rightarrow Tz = 0$. Since $z \in V$, there are scalars $a_1, \ldots a_n$ and $b_1, \ldots b_m$

such that $z = a_1 u_1 + \cdots + a_n u_n + b_1 v_1 + \cdots + b_m v_m$

Since $Tz = 0$ and T is linear:

$$0 = a_1 \underbrace{Tu_1}_{=0} + \cdots + a_n \underbrace{Tu_n}_{=0} + b_1 Tv_1 + \cdots + b_m Tv_m$$

$$\Rightarrow 0 = b_1 w_1 + \cdots + b_m w_m$$

But $(w_1, \ldots w_m)$ is linearly independent. Indeed, it is a subset of basis B_W, which is linearly independent (every subset of a linearly independent set is linearly independent)

$\Rightarrow b_1 = b_2 = \cdots = b_m = 0$

Therefore $z = a_1 u_1 + \cdots + a_n u_n \in U$

Thus $\text{null}(T) \subseteq U$.