Chapter 2

#14. U, W are 5-dimensional subspaces of $\mathbb{R}^9$. Prove $U \cap W \neq \{0\}$.

$\dim(U+W) \leq 9$ since $U+W$ is a subspace of $\mathbb{R}^9$

Also, $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

$= 5 + 5 - \dim(U \cap W)$

Thus, $9 \geq \dim(U+W) = 10 - \dim(U \cap W)$

$\Rightarrow \dim(U+W) \geq 1$, hence $U \cap W \neq \{0\}$.

Chapter 3

#2. Example of $f: \mathbb{R}^2 \to \mathbb{R}$ such that
$$f(av) = af(v), \forall a \in \mathbb{R}, \forall v \in \mathbb{R}^2$$
but f not linear.

Def: $f(x,y) = \begin{cases} \frac{y^2}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Then $f(a(x,y)) = f(ax, ay) = \begin{cases} \frac{a^2y^2}{ax} = a\frac{y^2}{x} = af(x,y) & \text{if } ax \neq 0 \\ 0 = af(x,y) & \text{if } ax = 0 \end{cases}$

But f is not linear: $f((0,1)+(1,1)) = f(1,2) = \frac{2^2}{1} = 4$
$$\neq$$
$$f(0,1) + f(1,1) = 0 + \frac{1^2}{1} = 1$$

#7 If $T \in \mathcal{L}(V,W)$ is surjective, and $\text{span}(v_1, \dots, v_n) = V$, then $(Tv_1, \dots, Tv_n)$ spans W.

Pf: Pick $w \in W$. We need to show that $w = c_1Tv_1 + \dots + c_nTv_n$ for some scalars $c_1 \dots c_n$.
Since T is surjective, there is $v \in V$ such that $Tv = w$ (*)
Now, $(v_1 \dots v_n)$ spans V. Thus, there exist scalars $c_1, \dots, c_n$ such that $v = c_1v_1 + \dots + c_nv_n \overset{(*)}{\Rightarrow} Tv = w = T(c_1v_1 + \dots + c_nv_n)$
Since T is linear, the last equation becomes $w = c_1Tv_1 + \dots + c_nTv_n$, which is what we wanted to show.