HW I (Chapter 1)

5 (a) $W_a = \{(x_1, x_2, x_3) \in F^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$ subspace of F^3?

Yes, $(0,0,0) \in W$

If (x_1, x_2, x_3) and $(y_1, y_2, y_3) \in W$, then
$$x_1 + 2x_2 + 3x_3 = 0$$
$$y_1 + 2y_2 + 3y_3 = 0$$
$$\xrightarrow{\text{add}} (x_1+y_1) + 2(x_2+y_2) + 3(x_3+y_3) = 0$$
so $(x_1+y_1, x_2+y_2, x_3+y_3) \in W$

$a(x_1, x_2, x_3) = (ax_1, ax_2, ax_3)$ satisfies $(ax_1) + 2(ax_2) + 3(ax_3) = 0$
so $a(x_1, x_2, x_3) \in W$

(b) No: $(4,0,0) \in W_b = \{(x_1, x_2, x_3) \in F^3 \mid x_1 + 2x_2 + 3x_3 = 4\}$
$(0,2,0) \in W_b$ but $(4,0,0) + (0,2,0) = (4,2,0) \notin W_b$
since $4 + 2 \cdot 2 + 3 \cdot 0 = 8 \neq 4$

(c) No: $(1,0,0)$ and $(0,1,1) \in W_c = \{(x_1, x_2, x_3) \in F^3 \mid x_1 x_2 x_3 = 0\}$
but $(1,0,0) + (0,1,1) = (1,1,1) \notin W_c$ since $1 \cdot 1 \cdot 1 = 1 \neq 0$

(d) Yes, proof similar as in (a)

9. Let U_1, U_2 be subspaces of V.S. V.

$U_1 \cup U_2$ is subspace of $V \iff U_1 \subseteq U_2$ or $U_2 \subseteq U_1$

Pf: $\Leftarrow$ obvious: If $U_1 \subseteq U_2$, then $U_1 \cup U_2 = U_2$, which is a subspace of V
(If $U_2 \subseteq U_1$, similar proof)

$\Rightarrow$ Suppose not. Then $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$. Thus, there exist
$u_1 \in U_1$ but $u_1 \notin U_2$ (*)
and $u_2 \in U_2$ but $u_2 \notin U_1$ (**)

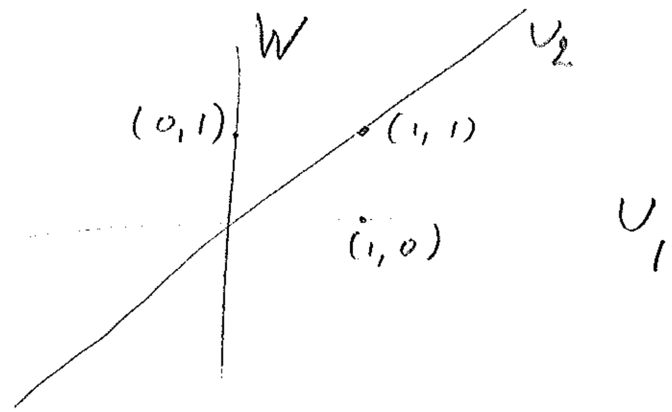
Thus, $u_1, u_2 \in U_1 \cup U_2$, hence as $U_1 \cup U_2$ is subspace,
also $u_1 + u_2 \in U_1 \cup U_2$. That is $u_1 + u_2 \in U_1$ or $u_1 + u_2 \in U_2$

- If $u_1 + u_2 \in U_1 \Rightarrow (u_1 + u_2) - u_1 \in U_1$ as U_1 is a subspace $\Rightarrow u_2 \in U_1$, contradicting (**)
- If $u_1 + u_2 \in U_2 \Rightarrow (u_1 + u_2) - u_2 \in U_2$ as U_2 is subspace $\Rightarrow u_1 \in U_2$, contradicting (*)

13. Prove or give counterexample:
U_1, U_2, W subspaces of V such that $U_1 + W = U_2 + W$. Then $U_1 = U_2$.

False: $V = \mathbb{R}^2$

$$U_1 = \operatorname{span}\{(1,0)\}$$
$$U_2 = \operatorname{span}\{(1,1)\}$$
$$W = \operatorname{span}\{(0,1)\}$$



Then $U_1 + W = U_2 + W = \mathbb{R}^2$ (Prove this !)

But $U_1 \neq U_2$

HW2 (Chapter 2)

3. Let $(v_1, \dots, v_n)$ be linearly independent in V, and $w \in V$.
Prove: If $(v_1 + w, \dots, v_n + w)$ is linearly dependent in V, then $w \in \operatorname{span}(v_1, \dots, v_n)$

Pf: Since $(v_1 + w, \dots, v_n + w)$ is linearly dependent, there are scalars $c_1, c_2, \dots, c_n$, not all 0, such that:

$$c_1(v_1 + w) + \dots + c_n(v_n + w) = 0.$$

$$\Rightarrow (c_1 + c_2 + \dots + c_n) w = -c_1 v_1 - c_2 v_2 - \dots - c_n v_n$$

- If $c_1 + c_2 + \dots + c_n \neq 0$, then $w = -\frac{c_1}{c_1 + \dots + c_n} v_1 - \dots - \frac{c_n}{c_1 + \dots + c_n} v_n \in \operatorname{span}(v_1, \dots, v_n)$
- If $c_1 + \dots + c_n = 0$, then $c_1 v_1 + \dots + c_n v_n = 0$, with the c_i's not all 0, which would imply that $(v_1, \dots, v_n)$ is linearly dependent, a contradiction

Thus, this case cannot occur.

6. Prove that $C[0,1] = \{f(x) \mid f:[0,1] \to \mathbb{R}, f \text{ is continuous}\}$ is infinite-dimensional

Pf: It suffices to define a sequence of functions $\{f_n\}$ in $C[0,1]$, such that for each m, the list $\{f_0, f_1, \dots, f_m\}$ is linearly independent. (See comment in margin on p 26)

Define $f_n(x) = x^n$, $n = 0, 1, 2, \dots$

Clearly each $f_n \in C[0,1]$.

Fix m. Need to show that $\{f_1, f_2, \dots, f_m\}$ is linearly independent:

Let $c_0 f_0 + \dots + c_m f_m = 0$ in $C[0,1]$,

That is $\underbrace{c_0 x^0 + c_1 x + c_2 x^2 + \dots + c_m x^m}_{\text{polynomial of degree } m} = \underbrace{0}_{\text{the zero function in } C[0,1]}$.

Let's call the left-hand side $g_m(x)$. Since this polynomial must be the zero polynomial, then must hold that:

$$g_m(0) = \frac{dg_m}{dx}(0) = \dots = \frac{d^m g_m}{dx^m}(0) = 0,$$

ie the value of g_m and its derivatives, evaluated at $x=0$, must all be 0.

This implies that $c_0 = c_1 = \dots = (m!)\,c_m = 0$

$\Rightarrow c_0 = c_1 = \dots = c_m = 0$

$\Rightarrow (f_0, \dots, f_m)$ is linearly independent

11. V finite-dimensional, $\dim V = n$.

There exist 1-dimensional subspaces $U_1 \dots U_2 \in V$:

$$V = U_1 \oplus \dots \oplus U_n$$

Pf: Since V is n-dimensional, it has a basis

$$\{v_1, v_2, \dots, v_n\}$$

Define $U_i = \operatorname{span}\{v_i\}$, $i = 1, \dots, n$

- Then $V = U_1 + \dots + U_n$. Indeed, every $v \in V$ can be written as $v = c_1 v_1 + \dots + c_n v_n$ for some scalars c_i since $(v_1 \dots v_n)$ is a basis for V.
  But this implies that $V \subseteq U_1 + \dots + U_n$
  That $U_1 + \dots + U_n \subseteq V$ is obvious:
  any linear combo $c_1 v_1 + \dots + c_n v_n$ belongs to V since V is a vector space with basis $(v_1 \dots v_n)$.

- To show that $V = U_1 \oplus \dots \oplus U_n$, it suffices to prove that the only way we can write the 0 vector as a sum of elements in the U_i, is by choosing all these elements 0 (See Prop 1.8)

$$\Rightarrow 0 = \overbrace{c_1 v_1}^{\in U_1} + \dots + \overbrace{c_n v_n}^{\in U_n}$$

But $(v_1 \dots v_n)$ is basis of V, hence $(v_1 \dots v_n)$ is linearly independent

$$\Rightarrow c_1 = c_2 = \dots = c_n = 0$$

$$\Rightarrow c_1 v_1 = c_2 v_2 = \dots = c_n v_n = 0 \text{, as required.}$$