Solutions to a couple of problems: MTG 3214*

October 29, 2008

Problem 1.8.5 on p.22 (assume problem 1.6.4 on p.18 has been solved; I'll do this later)

First a remark: The statement could have been more precise. For instance: "the 9 point circle cuts the sides at angles $|B-C|, |A-B|$ and $|A-C|$, seen from the points A, C and B respectively". An equivalent statement is: "the 9 point circle cuts the sides at angles $2|B-C|, 2|A-B|$ and $2|A-C|$, seen from its center N".

I will prove the last statement. Refer to Fig. 1.8B. The chord DA' is seen from N under angle:

$$\begin{aligned} \prec DNA' &= 2 \prec DKA' (\text{ since D,K,A' are on circle with center N}) \\ &= 2 \prec HKN (\text{ same angles }) \\ &= 2 \prec HAO (\text{ since } \triangle HKN \sim \triangle HAO, \text{ reason below}) \\ &= 2|B-C| (\text{ by problem 1.8.5}), \end{aligned}$$

There holds that $\triangle HKN \sim \triangle HAO$ by SAS, since $AH = 2KH$, the angles at H are the same, and $HO = 2HN$.

Problem 1.6.4 on p.18

Referring to Fig. 1.6A, we need to show that $\prec HAO = |B-C|$. We have

$$A = \prec BAC = \prec BAD + \prec HAO + \prec OAC. \tag{1}$$

We know that

$$\prec BAD = 90^o - B (\text{ consider the right-angled } \triangle BAD) \tag{2}$$

and that

$$\prec OAC = \prec OCA (\triangle AOC \text{ is isosceles because O is circumcenter}) = 90^o - B \tag{3}$$

where the last equality holds for a similar reason that in Fig. 1.6.A, $\prec OBC = \alpha = 90^o - A$ holds.

Plugging $(2), (3)$ in (1), and solving for $\prec HAO$, yields:

$$\prec HAO = A - 180^o + 2B = B - C,$$

which is $|B-C|$ since here $B > C$ (if B were $< C$ instead, you would find that $\prec HAO = C - B$, which is of course equal to $|B-C|$).

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