Problem for first homework assignment[1]

1 Background

Our genetic information is encoded in the *DNA* of our cells. This is merely a code, much like the code of a computer program, for the production of biological molecules, called *proteins*, that make up who we are. The conversion of DNA into protein happens in two steps. First it is converted into *mRNA*, another chemical substance, that contains the same genetic information as DNA does. This step is known as *transcription*. In the second step, called *translation*, mRNA is converted into protein. In short:

$$DNA \to mRNA \to protein.$$

2 Model

In the lab, we can interfere with the transcription process and measure the amounts of the various substances mentioned earlier. We will consider an experiment in which transcription is blocked. We shall write down a mathematical model that describes how the amounts of mRNA and protein vary with time. Let $x(t)$ denote the amount of mRNA and $y(t)$ the amount of protein at time t. Our model will consist of two differential equations, one for $x(t)$, and another for $y(t)$. It describes not only the translation process, but also the decay of mRNA and DNA.

The model is as follows:

$$\frac{dx}{dt}(t) = -d_1 x(t) \quad (1)$$

$$\frac{dy}{dt}(t) = -d_2 y(t) + kx(t) \quad (2)$$

The parameters d_1, d_2 and k are positive rates, and their unit is the inverse of the time unit (minutes or hours or seconds). The first equation implies that mRNA decays exponentially (just compare to radioactive decay which we discussed in class).

The second equation for protein is a bit more complicated. There are two terms on the right-hand side. The first term is a decay term: just like mRNA, the protein decays. The second term is a positive contribution that describes how fast protein is produced via transcription. Our model says that this rate is higher if there is more mRNA (because then $kx(t)$ is larger).

1. **Half-life**. Experimentalists often refer to the half-life of the substances they are studying. The half-life is defined as the time it takes in order that the amount of a decaying substance reaches half of its initial value.

   Assume first that $k = 0$. Show that the half-lifes of mRNA and protein are given by $\ln(2)/d_1$ and $\ln(2)/d_2$ respectively.

2. Assume that the half-lives of mRNA and protein are 14 and 16 hours respectively. Find d_1 and d_2. Using these values, first solve equation (1) with initial value $x(0) = 1$. Use your answer to solve equation (2), assuming that $k = 1$ per hour and $y(0) = 1$.

3 Afterthoughts

In practice, one of the purposes of a lab experiment could be to determine k, knowing d_1 and d_2 (or equivalently, the half-lives of mRNA and protein), and $x(0)$ and $y(0)$. Measurements taken during the experiment allow us to plot several data points which are simply values of the mRNA and protein at certain times. On the other hand, each choice for k yields graphs of the solutions $x(t)$ and $y(t)$ to equations (1) and (2). The goal is to find the value of k for which these graphs are the best fit (best, in a sense which can be made precise mathematically) to the data points obtained from the experiment.

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