Exam 1: MAP 2302*

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Name:
Student ID:

This is a **closed book** exam and the use of calculators is **not** allowed.

1. Solve the IVP:
$$\frac{dy}{dx} = -xy, \;\; y(1) = -1.$$

By the existence and uniqueness theorem, we know there is a unique solution. Separable equation:

$$\begin{aligned}
\frac{1}{y}dy &= -xdx, \\
\ln|y| &= -\frac{x^2}{2} + C, \; C \text{ arbitrary constant}, \\
|y| &= C'\mathrm{e}^{-\frac{x^2}{2}}, \; C' \text{ positive constant (since } C' = \mathrm{e}^C), \\
y &= \pm C'\mathrm{e}^{-\frac{x^2}{2}} = C''\mathrm{e}^{-\frac{x^2}{2}}, \;\; C'' \text{ nonzero constant}
\end{aligned}$$

Since $y(1) = -1$, it follows that

$$-1 = C''\mathrm{e}^{-\frac{1}{2}} \text{ or } C'' = -\mathrm{e}^{\frac{1}{2}},$$

and thus

$$y(x) = -\mathrm{e}^{\frac{1}{2}-\frac{x^2}{2}}$$

Common mistake:

After finding

$$|y| = C'\mathrm{e}^{-\frac{x^2}{2}},$$

one determines C', using the initial condition:

$$|-1| = C'\mathrm{e}^{-\frac{1}{2}}, \text{ so that } C' = \mathrm{e}^{\frac{1}{2}}.$$

So far, so good, but then one drops the absolute value and concludes:

$$y = \mathrm{e}^{\frac{1}{2}-\frac{x^2}{2}}.$$

This is wrong because $y(1) = +1$ instead of -1. The mistake made is that the absolute value cannot be dropped. A careful argument is as follows:

$$y = \pm\mathrm{e}^{\frac{1}{2}-\frac{x^2}{2}}.$$

These are two functions $y(x)$, while we know there should only be one (by the existence and uniqueness theorem). So which one is the solution to our problem? That's easy: the one corresponding to the $-$ part since it satisfies the initial condition $y(1) = -1$. The other function does not.

*Instructor: Patrick De Leenheer.

2. Solve the following IVP for all $t \geq 0$:

$$\frac{dx}{dt} - x = u(t), \; x(0) = 0,$$

where

$$u(t) = \begin{cases} 1, \; t \in [0,1] \\ 0, \; t > 1 \end{cases}$$

Sketch the solution $x(t)$ in the (t,x) plane for $t \geq 0$.
What is the value of $x(t)$ at $t = 1$?

Linear equation with integrating factor:

$$\mu(t) = \mathrm{e}^{\int -1 dt} = \mathrm{e}^{-t}.$$

It follows that

$$\frac{d}{dt}\left(\mathrm{e}^{-t}x(t)\right) = \mathrm{e}^{-t}u(t),$$

and thus by integrating from 0 to t that:

$$\mathrm{e}^{-t}x(t) - \mathrm{e}^{0}x(0) = \int_0^t \mathrm{e}^{-\tau}u(\tau)d\tau$$

Since $x(0) = 0$ we find that:

$$x(t) = \mathrm{e}^{t}\int_0^t \mathrm{e}^{-\tau}u(\tau)d\tau$$

Case 1: $t \in [0,1]$, so $u(t) = 1$.

$$x(t) = \mathrm{e}^{t}\int_0^t \mathrm{e}^{-\tau}d\tau = \mathrm{e}^{t}(-\mathrm{e}^{-t}+1) = \mathrm{e}^{t} - 1.$$

In particular,

$$x(1) = \mathrm{e}^{1} - 1.$$

Case 2: $t > 1$, so $u(t) = 0$ (although of course still $u(t) = 1$ for $t \in [0,1]$).

$$\begin{aligned} x(t) &= \mathrm{e}^{t}\left(\int_0^1 \mathrm{e}^{-\tau}d\tau + \int_1^t 0 d\tau\right) \\ &= \mathrm{e}^{t}(1-\mathrm{e}^{-1}) \end{aligned}$$

Summarizing:

$$x(t) = \begin{cases} \mathrm{e}^{t} - 1, \; t \in [0,1] \\ \mathrm{e}^{t}(1-\mathrm{e}^{-1}), \; t > 1 \end{cases}$$

Common mistake:

Instead of integrating from 0 to t, one could also perform an indefinite integration:

$$\mathrm{e}^{-t}x(t) = \int \mathrm{e}^{-\tau}u(\tau)d\tau.$$

There is nothing wrong with this approach..

Case 1: $t \in [0,1]$

$$\mathrm{e}^{-t}x(t) = \int \mathrm{e}^{-\tau}u(\tau)d\tau = \int \mathrm{e}^{-\tau}d\tau = -\mathrm{e}^{-t} + C,$$

and since $x(0) = 0$ we find that

$$0 = -1 + C, \; \text{or } C = 1.$$

Thus,

$$x(t) = \mathrm{e}^t(-\mathrm{e}^{-t} + 1) = -1 + \mathrm{e}^t.$$

Case 2: $t > 1$.

$$\mathrm{e}^{-t}x(t) = \int \mathrm{e}^{-\tau}u(\tau)d\tau = \int 0 d\tau = C',$$

Most people now use $x(0) = 0$ to determine C', but this is wrong. The initial condition $x(0) = 0$ holds for $t = 0$, while we are considering the case where $t > 1$. To fix this, use as initial condition $x(1) = -1 + \mathrm{e}^1$ which can be determined from **Case 1**. Plugging this into previous equation, we find that:

$$\mathrm{e}^{-1}(-1 + \mathrm{e}^1) = C' \text{ or } C' = 1 - \mathrm{e}^{-1}.$$

Finally, for $t > 1$ we have that

$$x(t) = \mathrm{e}^t \left(1 - \mathrm{e}^{-1}\right)$$

Clearly, both approaches give the same result.

3. Classify each of the following equations as separable (S), linear (L), exact (E), homogeneous (H) or of Bernoulli type (B). In case of Bernoulli type, say what n is.

Equation	S	L	E	H	B
$(x + y)dx + (x - y)dy = 0$	NO	NO	YES	YES	NO
$\sin(t)\frac{dx}{dt} + x\cos(t) = \tan(t)$	NO	YES	YES	NO	YES, $n = 0$

4. Solve:

$$\frac{dy}{dx} = \frac{x^2 + y^2 + xy}{x^2}, \quad x > 0.$$

This is Example 1 of Section 2.6 (p. 74). The solution can be found there.