Notes on matrices and discrete systems

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September 3, 2007

Abstract

These notes complement section 9.1 to 9.3 in our Nagle/Saff/Snider text, and illustrate the different concepts from linear algebra described there in the context of discrete time linear systems. If you have no, or limited background in linear algebra, you should carefully review both these notes and the sections from the book.

1 A motivating example

Let's consider a model of some population with 2 stages, a juvenile and an adult stage. Let $J(t)$, $M(t)$ denote the number of juveniles and adults at time t. Since we only take a census of the population every year (or month, or day), time is discrete here, so $t = 0, 1, 2, 3, \ldots$ We will assume that juveniles turn into adults and that adults generate offspring. Our first goal is to make a precise mathematical model that describes this process. Secondly, we would like to use that model to make predictions about the future composition of the population given the initial composition.

Assumptions for model:

1. We assume that none of the juveniles or adults die.
2. Let f be the fraction of juveniles that matures to adulthood by the next census. Then $1-f$ is the fraction of juveniles that remains in the juvenile class.
3. Once an individual becomes an adult, it remains an adult.
4. Let m be the number of offspring generated by 1 adult in 1 unit of time.

Of course, we could immediately criticize this model. For instance, it is quite silly to assume that no individual ever dies. For now, we will ignore this, and later we will actually modify our model to reflect death.

Question: Given the current composition of the population, what is the composition at the time of the next census? In other words, knowing $J(t)$ and $M(t)$, what will $J(t+1)$ and $M(t+1)$ be? Well, $fJ(t)$ of juveniles grow into adults and all $M(t)$ adults remain adult, so

$$M(t+1) = fJ(t) + M(t).$$

Similarly, $(1-f)J(t)$ juveniles remain juveniles while $mM(t)$ offspring are born which enter the juvenile class, so

$$J(t+1) = (1-f)J(t) + mM(t)$$

So our model, which holds for all $t = 0, 1, 2, \ldots$ is:

$$\begin{aligned} J(t+1) &= (1-f)J(t) + mM(t) \\ M(t+1) &= fJ(t) + M(t) \end{aligned}$$

But what does all this have to do with matrices? Well, it turns out that it is very useful to compactify the above model by defining certain vectors (which is just a column of numbers) and a matrix (which is really just a table of numbers). Let

$$x(t) = \begin{pmatrix} J(t+1) \\ M(t+1) \end{pmatrix} \text{ and } M = \begin{pmatrix} 1-f & m \\ f & 1 \end{pmatrix}$$

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Then $x(t)$ is a two-dimensional vector and M is a 2×2 matrix. Formally, $x(t) \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$. Our population model can now be written as follows:

$$x(t+1) = Ax(t),$$

quite a bit of savings of our valuable ink, no? Of course, we have not made clear what $Mx(t)$ is. It is a product of a matrix and a vector, so we should explain this a bit.

2 Matrices and vectors

See Section 9.3 in the book. A matrix is nothing more than a table of (complex or real, but we'll assume complex here) numbers. If matrix A has n rows and m columns, we say that A is in $\mathbb{C}^{n\times m}$. A column vector with n entries, like $x(t)$ above, can be considered as a specific matrix, namely a $n \times 1$ matrix, while a row vector with n entries is an $1 \times n$ matrix. There are two basic operations with matrices:

1. Multiplication by a scalar. Example:

$$3\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$$

2. Addition. You can only add matrices with the **same dimensions**. Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

3. Multiplication. You can only multiply matrices with **compatible dimensions**, namely AB makes sense only if $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times p}$. In other words, A should have exactly as many columns as B has rows, namely m. If this is the case, then $AB \in \mathbb{C}^{n\times p}$ and the (i,j)-th entry of AB is **the dot product of row i of A and column j of B** (check your old calc book in case you forgot about dot products of vectors). Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

3 Discrete time linear systems

We want to generalize our particular population model from above a bit to encompass other sorts of systems. Let $x(t)$ be an n-dimensional column vector, describing the **state** of some system at time t. The system we are considering here could be very different from our population model. It could represent a mechanical system where the state vector contains the positions and velocities of a bunch of particles moving through space. Or we could be dealing with some electrical circuit were the components of the state vector represent currents in some wires or voltages over certain electrical components in the circuit. Generally, we define a **discrete time linear system** as follows:

$$x(t+1) = Ax(t). \tag{1}$$

The terminology stems from the fact that (1) can be interpreted as a linear map from a vector space $\mathbb{R}^n$ to itself, mapping a vector $x(t)$ to a vector $x(t+1)$. Recall from linear algebra that when V and W are vector spaces, then a map $T : V \to W$ is called **linear** if $T(ax+by) = aT(x)+bT(y)$ for all vectors x, y in V and all scalars a and b. Convince yourself now that the map taking a vector x in $\mathbb{R}^n$ to the vector Ax in $\mathbb{R}^n$, is indeed linear. This explains the terminology.

So in practice, calculating $x(t+1)$ from $x(t)$, amounts to a simple matrix multiplication! Let's illustrate this by our population example. Suppose we start with an initial (at $t = 0$) population having a composition of 100 juveniles and 0 adults. Then at time 1, the composition will be:

$$x(1) = \begin{pmatrix} 1-f & m \\ f & 1 \end{pmatrix}\begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} 100(1-f) \\ 100f \end{pmatrix},$$

at time $t = 2$, it is:

$$x(2) = \begin{pmatrix} 1-f & m \\ f & 1 \end{pmatrix} x(1) = \begin{pmatrix} 100(1-f)^2 + 100mf \\ 100(2f - f^2) \end{pmatrix},$$

and so on for $x(3)$, $x(4)$, ...

Clearly, for our general discrete time linear system (1), we can write down the value of the state at any given time $t = 0, 1, 2, ...$, given the initial state $x(0)$[1]:

$$x(t) = A^t x(0), \tag{2}$$

where A^t of course denotes the product of the matrix A with itself, t times (we agree that A^0 is the so-called **identity matrix** I, which has 1's on its diagonal and 0's elsewhere).

So this is one particular illustration of why matrices are useful. After some reflection, we realize that in general, multiplying a matrix t times with itself, becomes a quite tedious, computationally intensive task. It turns out that using some results from linear algebra, there is a very elegant way to quickly compute an arbitrary power of a matrix.

4 Matrix powers

Let's start this topic by pointing out a very special, yet very important case. If $A \in \mathbb{C}^{n\times n}$ is a **diagonal** matrix, that is to say, it has the following structure:

$$A = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix},$$

then calculating A^t for any $t = 0, 1, 2, 3...$ is child's play. The power A^t is also diagonal:

$$A^t = \begin{pmatrix} d_1^t & 0 & \dots & 0 \\ 0 & d_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n^t \end{pmatrix}$$

Now recall a definition from linear algebra: A matrix $A \in \mathbb{C}^{n\times n}$ is **diagonalizable** if there is some matrix $T \in \mathbb{C}^{n\times n}$ which is invertible[2], such that

$$T^{-1}AT = D,$$

where D is some diagonal matrix.

So why would this help to calculate A^t? Here's why:

$$A^t = (TDT^{-1})^t = (TDT^{-1})(TDT^{-1})\dots(TDT^{-1}) = TD^tT^{-1}, \tag{3}$$

and since D^t is diagonal, the final matrix, TD^tT is very easily calculated as the product of three matrices! By the way, you should carefully check the above calculations and see where we used associativity of matrix multiplication, the property that $TT^{-1} = T^{-1}T = I$, and the property that multiplying an arbitrary square matrix by I on either left or right, yields that matrix again.

But is it really that easy? This would mean that computing A^{100} for some 100×100 matrix amounts to multiplying just 3 matrices of size 100×100? So instead of multiplying 99 times we only need to multiply 2 times? Surely, there must be some catch here, and we must be overlooking something!? Unfortunately we are: given an arbitrary matrix A, it is not so clear how to find the matrices T and D; even worse, it's not even clear whether or not such a matrices exist! In fact, finding an answer to this question is one of the main goals in a typical linear algebra course.

Theorem 1. *A matrix $A \in \mathbb{C}^{n\times n}$ is diagonalizable if and only if it has a basis of eigenvectors.*

Although this result looks very short and neat, it makes no sense to us right now since we have not said what an eigenvector is or what a basis is. So, let's take a deep breath, and get ready to make a serious detour first, before proving this important result.

[1] Can you prove this claim, using induction?

[2] A matrix $T \in \mathbb{C}^{n\times n}$ is invertible if there is some matrix $T^{-1} \in \mathbb{C}^{n\times n}$ such that $TT^{-1} = T^{-1}T = I$.

4.1 Basis of a vector space

Definition 1. *Let V be a vector space, and $S = \{v_1, v_2, \dots, v_m\}$ be a subset of vectors in V. We say that S is linearly dependent if there are scalars $c_1, c_2, \dots, c_m$,* **not all zero**, *such that*

$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = 0.$$

The set S is linearly independent if it is not linearly dependent. In other words, S is linearly independent if the following holds:

If $c_1, \dots, c_m$ are such that $c_1 v_1 + \cdots + c_m v_m = 0$, then necessarily $c_1 = c_2 = \cdots = c_m = 0$.

Examples:

1. In $\mathbb{R}^2$ the set
$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$
is linearly dependent since we can pick $c_1 = 2$ and $c_2 = -1$, and these scalars are not both zero (in fact, both are nonzero).

2. In $\mathbb{R}^2$ the set
$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$
is linearly independent. To see this assume that c_1 and c_2 are such that
$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
Equivalently,
$$\begin{aligned} c_1 + c_2 &= 0 \\ c_2 &= 0, \end{aligned}$$
and thus necessarily
$$c_1 = c_2 = 0.$$

Exercises:

1. Let a be a real number. Give a condition for a such that
$$\left\{ \begin{pmatrix} 1 \\ a \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\}$$
is linearly independent in $\mathbb{R}^2$.

2. Let S be a finite subset of V that contains the zero vector. Show that S is linearly dependent in V.

Definition 2. *Let $G = \{g_1, g_2, \dots, g_p\}$ be a subset of a vector space V. We say that G is a generating set for V if every $v \in V$ can be written as a linear combination of elements in W, i.e. for any given $v \in V$ there are scalars $c_1, \dots, c_p$ such that*

$$v = c_1 g_1 + c_2 g_2 + \cdots + c_p g_p.$$

Example: The set
$$W = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$
is not generating for $\mathbb{R}^2$ since for instance the vector
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
can not be written as a linear combination of vectors in S.

Definition 3. *Let $B = \{b_1, b_2, \dots, b_n\}$ be a subset of a vector space V. Then B is called a basis for V if B is linearly independent and generating for V.*

Example: The vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

are a basis for $\mathbb{R}^n$, and also for $\mathbb{C}^n$.[3] This basis is called the standard basis, and its vectors the standard basis vectors.

Now that we know what a basis is, it's time to explain what an eigenvector is.

4.2 Eigenvalues and eigenvectors

Definition 4. *Let $A \in \mathbb{C}^{n\times n}$. We say that the vector $x \in \mathbb{C}^n$ is an eigenvector of A if $x \neq 0$, and if there is some scalar $\lambda \in \mathbb{C}$, such that*

$$Ax = \lambda x. \tag{4}$$

The scalar λ is called the eigenvalue corresponding to the eigenvector x.

It's important to note that eigenvectors **are not be zero**! How do we determine these eigenvector-eigenvalue pairs? Let's examine (4), and re-write the problem a bit. We are trying to find a nonzero $x \in \mathbb{R}^n$ such that there is some scalar λ such that:

$$(A - \lambda I)x = 0. \tag{5}$$

This means that we are asking that the kernel[4] of the matrix $A - \lambda I$ contains a non-zero vector. Notice that this is equivalent with saying that the columns of the matrix are linearly dependent. In linear algebra it is shown that this is equivalent with saying that the matrix $A - \lambda I$ is not invertible. We then call the matrix **singular**.

It turns out that for arbitrary square matrices, there is a simple test to check if they are singular:

The matrix $B \in \mathbb{C}^{n\times n}$ is singular if and only if its determinant $\det(B)$, is zero.

The topic of determinants is worth a study in itself, but for the purposes of this course, it will be enough that you are able to calculate it for 2×2 and 3×3 matrices. Basically, given a matrix, its determinant is a scalar which you compute as follows:

1. For matrices in $\mathbb{C}^{2\times 2}$. Let
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$
then
$$\det(A) = a_{11}a_{22} - a_{21}a_{12}$$
So, it is the difference of the product of the diagonal entries and the product of the off-diagonal entries.

2. The determinant of matrices in $\mathbb{C}^{3\times 3}$ is given in terms of the determinant of matrices in $\mathbb{C}^{2\times 2}$. Let
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$
then
$$\det(A) = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

[3] Prove this

[4] The kernel of a matrix $B \in \mathbb{C}^{n\times m}$ is defined as the set of all vectors $x \in \mathbb{C}^m$ for which $Bx = 0$.

Examples:

1.
$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 - 6 = 3.$$

2.
$$\det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{pmatrix} = (9-8) - 2(6-6) + (8-9) = 0,$$

so this matrix is singular, which we anticipated since the first and last column are the same, so the columns are linearly dependent.

Returning to our eigenvector-eigenvalue problem, let's see what happens if A is a 2×2 matrix. Then writing $\det(A - \lambda I) = 0$ more explicitly, we have:

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0.$$

After some manipulations, we find that λ (unknown for now!) should satisfy

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0.$$

In other words, the λ's we're after, are roots of a quadratic equation, which we call the **characteristic equation** of the matrix A.

Exercise: Determine the characteristic equation of a matrix in $\mathbb{C}^{3\times 3}$. It turns out that it is a cubic equation.

This can be generalized to arbitrary square matrices:

If $A \in \mathbb{C}^{n\times n}$, then its characteristic equation is

$$p_n(\lambda) = 0,$$

where $p_n(\lambda)$ is an n-th order polynomial whose coefficients depend on the entries of A.

Consequently, and more importantly,

Eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$ are the roots of an n-th order polynomial!

The Fundamental Theorem of Algebra tells us that an n-th order polynomial with complex coefficients has n roots (some of which may coincide of course), and thus a matrix $A \in \mathbb{C}^{n\times n}$ has n eigenvalues. Once you have found the eigenvalues of a matrix, you determine their eigenvectors as follows: Let λ be an eigenvalue. Plug it into (5), and solve the resulting set of linear equations for some nonzero x. An example will make things clear.

Example: Let
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

then its characteristic equation is

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = 0,$$

which has 2 roots
$$\lambda_1 = 0, \ \ \lambda_2 = 2.$$

To find an eigenvector associated to λ_1, replace λ in (5) by 0 to get

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or more explicitly

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_2 &= 0, \end{aligned}$$

which is satisfied when we pick $x_1 = 1$ and $x_2 = -1$ for instance (Remember, we're only interested in a nonzero solution, that is either x_1 or x_2 should be nonzero). So

$$x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is and eigenvector associated to the eigenvalue $\lambda_1 = 0$. Of course we could have picked $x_1 = a$ and $x_2 = -a$, for any given nonzero scalar a! In fact, this is a general property of eigenvectors: If x is an eigenvector associated to eigenvalue λ, then ax is also an eigenvector associated to the same eigenvalue λ, for arbitrary nonzero scalars a.[5]

Exercise: Show that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector associated to eigenvalue $\lambda_2 = 2$.

4.3 Proof of Theorem 1

Only if-part Let A be diagonalizable. Then there is a non-singular matrix $T \in \mathbb{C}^{n\times n}$ such that

$$T^{-1}AT = D,$$

and thus

$$AT = TD.$$

More explicitly, there are column vectors $t_1, \dots, t_n$ (namely the columns of the matrix T) such that

$$At_i = d_i t_i,$$

for all $i = 1, 2, \dots, n$. In other words, the t_i are eigenvectors of A with corresponding eigenvalues d_i. Now since the matrix T is non-singular, its columns form a basis for $\mathbb{C}^{n\times n}$[6]

If-part Let x_i, $i = 1, 2, \dots, n$ be a basis of eigenvectors of A, each having a corresponding eigenvalue λ_i. Then

$$Ax_i = \lambda_i x_i,$$

for all $i = 1, \dots, n$. But using matrix notation, this is the same as writing

$$AX = X\Lambda,$$

where

$$X = [x_1 \;\; x_2 \;\; \dots \;\; x_n] \text{ and } \Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Since the columns of X form a basis for $\mathbb{C}^{n\times n}$, it follows that X is invertible, and hence

$$X^{-1}AX = \Lambda,$$

so that A is indeed diagonalizable.

Discussion

This theorem is important for 2 reasons:

1. It tells us exactly which matrices are diagonalizable, namely those that have a basis of eigenvectors.

2. It helps us in our initial calculation problem of powers of a matrix, by providing us with the matrices T and D. Indeed, the above proof reveals that T is simply the matrix whose columns are the n eigenvectors of A, and the matrix D is the diagonal matrix whose diagonal elements are just the eigenvalues of the matrix A.

[5] Prove this.

[6] Here, we used the -unproved- fact from linear algebra that if a set $S = \{v_1, v_2, \dots, v_n\}$ of n vectors in $\mathbb{C}^{n\times n}$ is linearly independent, then it is also generating for $\mathbb{C}^n$, and hence forms a basis for $\mathbb{C}^{n\times n}$.

So really, we just killed two birds with one stone!

Before returning to our population model, let us point out that

1. Some matrices are *not diagonalizable*! The most important example to keep in mind is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   Exercise: Verify that this matrix is not diagonalizable. This is an example of a matrix which is in **Jordan canonical form**. It is also an example of a so-called **nilpotent matrix** (A matrix $B \in \mathbb{C}^{n\times n}$ is nilpotent if there is some positive integer k such that A^k is the zero matrix). Later in our course, such matrices may play an important role, but for now, they don't. Just be aware of their existence.

2. In linear algebra it is shown that there is an important class of matrices which are always diagonalizable, namely matrices that have **distinct eigenvalues**. Keep this in mind, as it arises very often in applications.

3. Another important class of matrices which are always diagonalizable are **symmetric matrices** in $\mathbb{R}^{n\times n}$. A matrix A in $\mathbb{R}^{n\times n}$ is symmetric if the (i,j)th entry equals the (j,i)th entry of the matrix for all $i, j = 1, \ldots, n$. More information is known about the eigenvalues and eigenvectors of symmetric matrices. This is an important result from linear algebra which we reproduce here without proof.

   Theorem 2. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Then it has real eigenvalues and it is diagonalizable. Moreover, the eigenvectors corresponding to the eigenvalues form an orthogonal basis of $\mathbb{R}^n$ (that is, the dot product of two distinct eigenvectors is zero). In other words, when*

$$T^{-1}AT = D,$$

   then D has real entries, and distinct columns of the matrix T are orthogonal (any matrix with this property is called an orthogonal matrix).

5 Population model revisited

Let $f = 1$ so that all juveniles mature into adults in 1 time step, and $m = 2$, so that an adult generates 2 offspring in that same period. Assume that we start with 10 juveniles and 15 adults.

Question Calculate the composition of the population for arbitrary t.

The answer is

$$x(t) = A^t x(0),$$

where

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \text{ and } x(0) = \begin{pmatrix} 10 \\ 15 \end{pmatrix}.$$

From our discussion on matrix powers we know that IF A is diagonalizable, then

$$A^t = TD^tT^{-1},$$

where D is a diagonal matrix with the eigenvalues of A and the columns of T are the corresponding eigenvectors. We will see shortly that A is indeed diagonalizable. The eigenvalues of A are roots of (check this!)

$$\lambda^2 - \lambda - 2 = 0,$$

and they are

$$\lambda_1 = 2, \;\; \lambda_2 = -1,$$

with corresponding eigenvectors

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \;\; x_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

Since $\{x_1, x_2\}$ for a basis for $\mathbb{R}^2$ if follows from Theorem 1 that A is diagonalizable. Let

$$T = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then by the subsequent remark

$$T^{-1} = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}.$$

We conclude that

$$x(t) = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} D^t \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 10 \\ 15 \end{pmatrix} = 2^t \begin{pmatrix} 40/3 \\ 40/3 \end{pmatrix} + (-1)^t \begin{pmatrix} -10/3 \\ 5/3 \end{pmatrix}.$$

Notice that although the expression for the solution $x(t)$ contains fractional terms, it will always be a vector with integer-valued components, when $x(0)$ has integer-valued components. Moreover, even though negative terms appear in $x(t)$ (namely at odd values of time t), the components of $x(t)$ are never negative, as they should, since they represent numbers of individuals that are in a certain stage.

Remark 1. We did not say yet how to compute the inverse of an invertible matrix. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be invertible (i.e. $\det(A) = ad - bc \neq 0$), then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For an invertible 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

we have that

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} +A_{11} & -A_{21} & +A_{31} \\ -A_{12} & +A_{22} & -A_{32} \\ +A_{13} & -A_{23} & +A_{33} \end{pmatrix}$$

(notice the inverted indices!!!), where for every $i, j = 1, 2, 3$, each A_{ij} is the determinant of the matrix that you obtain by deleting the ith row and jth column from the matrix A. For example,

$$A_{23} = \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}.$$

6 Stability concepts

Let us start with a very simple case that reveals the key issues concerning the notion of stability. Consider a 1-dimensional discrete time linear system:

$$x(t+1) = ax(t), \quad t = 1, 2, \ldots$$

for some $a \in \mathbb{R}$. The system is 1-dimensional since $x(t)$ takes values in $\mathbb{R}^1 \equiv \mathbb{R}$. Given the initial condition $x(0) = x_0$ we wonder what happens to the solution sequence $x(1), x(2), \ldots$ when $t \to \infty$. Does the sequence converge, or not? Does it remain bounded, or not? Well, let's see. We know that

$$x(t) = a^t x_0, \quad t = 0, 1, \ldots,$$

and therefore,

1. If $|a| < 1$, then $\lim_{t\to\infty} x(t) = 0$, no matter what x_0 is.

2. If $|a|$=1, then $x(t)$ remains bounded for all t, no matter what x_0 is. In fact, if $a = 1$, then $x(t) = x_0$ for all t, whereas if $a = -1$, then $x(t) = (-1)^t x_0$, so the solution sequence oscillates, or jumps, between x_0 and $-x_0$.

3. If $|a| > 1$, then $\lim_{t\to\infty} x(t) = \infty$ when $x_0 \neq 0$. In particular, if a solution does not start in 0, it grows unbounded (of course, if it did start in 0 it remains there forever after).

So it turns out that the deciding factor in this discussion is whether or not the **absolute value of a is less than or greater than** 1. We would like to extend this to general discrete time linear systems (1). If we are really cavalier and decide to jump in blindly, we would be tempted to say that it would depend on the absolute value of the matrix A. But what is the absolute value of a matrix? Hmm, not so clear? We will show in a few minutes that we should phrase our condition in terms of **the moduli of the eigenvalues of the matrix** A. The word "modulus"[7] replaces the word "absolute value" since for general matrices A -even those having only real entries- the eigenvalues could be complex numbers (for example, check that the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has eigenvalues $\pm i$).

OK, enough philosophy, let's make all this mathematically precise. Consider (1) and assume that the matrix A is diagonalizable. By Theorem 1 we know that A has a basis of eigenvectors, let's call them $v_1, v_2, \dots, v_n$ with associated eigenvalues $\lambda_1, \lambda_1, \dots, \lambda_n$. Now consider the initial condition x_0 for our system. Since the eigenvectors of A form a basis for $\mathbb{R}^n$, we can write

$$x_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \tag{6}$$

for appropriately chosen (complex) scalars $c_1, c_2, \dots, c_n$ (recall that we can always do this because a basis is a generating set). Consider the solution sequence that starts in x_0. At time t it is

$$x(t) = A^t x_0,$$

and combining this with (6), we find that

$$x(t) = A^t(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1A^tv_1 + c_2A^tv_2 + \cdots + c_nA^tv_n = c_1\lambda_1^tv_1 + c_2\lambda_2^tv_2 + \dots c_n\lambda_n^tv_n.$$

In the last step we used that $Av_i = \lambda_i v_i$ for all $i = 1, 2, \dots, n$. From this expression we immediately see that

Theorem 3. *Let $A \in \mathbb{R}^{n\times n}$ be diagonalizable, with eigenvalue-eigenvectors pairs (λ_i, v_i) for $i = 1, \dots, n$.*

1. *If $|\lambda_i| < 1$ for all $i = 1, \dots, n$, then $\lim_{t\to} x(t) = 0$, no matter what x_0 is.*

2. *If $|\lambda_i| \leq 1$ for all $i = 1, \dots, n$, then $x(t)$ remains bounded.*

3. *If there is some eigenvalue λ_j such that $|\lambda_j| > 1$, then $x(t)$ grows unbounded for almost all initial conditions x_0. More precisely, $x(t)$ grows unbounded whenever the initial condition x_0 is such that $c_j \neq 0$ in (6).*

Visualizing this in the complex plane $\mathbb{C}$, this amounts to checking whether or not the eigenvalues of A are inside (case 1), or on (case 2) the unit circle $S = \{z \in \mathbb{C} \mid |z| = 1\}$, or that A has an eigenvalue outside the unit circle (case 3).

Remark 2. Although we will postpone a precise definition of the notions, stability, asymptotic stability and instability, we say that system (1) **is asymptotically stable in case** 1**, stable in case** 2 **and unstable in case** 3. This terminology is not entirely precise, as these notions should be defined for particular solutions of system (1), not for the system itself. But when no reference is made to a particular solution, then one is usually implicitly assuming that the zero solution is meant. So to be precise, we should say that the zero solution of system (1) is stable, asymptotically stable or unstable.

[7] Recall that the modulus of a complex number $z = a + ib$ is defined as $|z| = \sqrt{a^2 + b^2}$.

Remark 3. The previous remark suggests that there may be other particular solutions aside from the zero solution. A point $z \in \mathbb{R}^n$ is called a **fixed point** of system (1) if $z = Az$. Clearly, the terminology stems from the fact that if we pick an initial condition in a point that satisfies this equation, then the solution sequence is the constant sequence, $z, z, z, \dots$ In other words, the solution remains in z for all subsequent times. Notice that $z = 0$ is always a fixed point of system (1), which explains why stability of a system is sometimes equated with stability of the zero solution as we just explained. But what about nonzero fixed points?

Exercise. Show that system (1) has a nonzero fixed point z if and only if z is an eigenvector of the matrix A with associated eigenvalue $\lambda = 1$.

7 Discrete time Markov chains

In this section we discuss an important application of discrete time linear systems, namely discrete time Markov chains. But let's begin modestly, and consider a simple computer network with 2 PC's, and assume we wish to study an information packet -like a file- that travels between the PC's. Unfortunately, generally we don't know where the packet is, unless perhaps at the initial time. For subsequent times, the only information available to us is the probability that the packet is in PC1. (and then 1 minus that probability is the probability that the packet is in PC2). Thus, the state of our system is the following vector

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

where $x_i(t)$ denotes the probability that the packet is in PCi at time t, for $i = 1$ or 2. Of course, $x_1(t) + x_2(t) = 1$, while $x_i(t) \in [0, 1]$ for $i = 1, 2$. Now, assuming that the packet is in PC1 at a certain time, then the probability that in the subsequent time, the packet has switched to PC2 is p_{21}. Similarly, if it was at PC2, then the probability it has switched to PC1 is p_{12}. Consequently, if it was at PC1, then the probability that the packet is still at PC1 at the subsequent time, is $1 - p_{21}$ and similarly, the probability that is stays at PC2 when it starts there, is $1 - p_{12}$. We can now write down equations for $x(t + 1)$, given $x(t)$:

$$\begin{aligned} x_1(t+1) &= (1 - p_{12})x_1(t) + p_{12}x_2(t) \\ x_2(t+1) &= p_{21}x_1(t) + (1 - p_{21})x_2(t) \end{aligned}$$

In matrix notation,

$$x(t + 1) = Px(t),$$

where

$$P = \begin{pmatrix} 1 - p_{21} & p_{12} \\ p_{21} & 1 - p_{12} \end{pmatrix} \tag{7}$$

This matrix P is an example of a so-called **stochastic matrix** since all its entries are in $[0, 1]$, and the entries in each column add up to 1.

Of course, nowadays, most computer networks are much larger than that. But that is alright as we can easily extend our example to model large networks. First, let us slightly modify the problem setting. Instead of tracking the probability that some information packet is at a certain PC, the state of our system will consist of the probabilities that a websurfer is visiting a certain website of one of the n websites on the World Wide Web. Thus, let $x_i(t)$ for $i = 1, \dots, n$ be the probability that a websurfer is visiting website i at time t, and assume that the probability that he jumps from site i to site j in two subsequent times is p_{ji}[8] (and the probability that he stays at site i is given by $p_{ii} = 1 - \sum_{k \neq i} p_{ki}$). These probabilities are called the **transitional probabilities** of the Markov chain. The model becomes

$$x(t + 1) = Px(t), \tag{8}$$

[8] Our notation deviates from the standard one in the Markov chain literature where this probability is denoted by p_{ij} instead.

where

$$P = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{pmatrix} \tag{9}$$

Then clearly each entry of the matrix is in $[0,1]$ and the entries in each column add up to 1, so that P is indeed a stochastic matrix.

Let us check some obvious properties of system (8). Since the state $x(t)$ represents a probability vector, its components should be in $[0,1]$, and they should add up to 1 at every time if the initial state $x(0)$ has the same property. That the components of $x(t)$ remain non-negative for all t is obvious when $x(0)$ is a probability vector since the matrix P has only non-negative entries. So it suffices to verify that the components of every $x(t)$ add up to 1, and by induction it is enough to do this for 1 time step:

$$x_1(1) + x_2(1) + \cdots + x_n(1) = (1\ 1\ \dots\ 1)Px(0) = (1\ 1\ \dots\ 1)x(0) = 1,$$

where we used the fact that the columns of P add up to 1.

Question: What can be said about the probability vector $x(t)$ when $t \to \infty$?

From before we know that the eigenvalues of P will play an important role. What can be said about these eigenvalues? First, let's use matrix notation to express that the columns of P add up to 1:

$$(1\ 1\ \dots\ 1)P = (1\ 1\ \dots\ 1),$$

or equivalently, that

$$P^T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

where P^T denotes the **transpose of the matrix** P (this is the matrix obtained from P by replacing the (i,j)th entry of P by its (j,i)th entry, for all $i, j = 1, 2, \dots, n$). It is shown in linear algebra -no proof here- that the eigenvalues of any matrix and its transpose are the same! This implies therefore that 1 is an eigenvalue of the matrix P. In fact, using the so-called Perron-Frobenius Theorem from linear algebra (not proved here) it can be shown that the modulus of all other eigenvalues of P are not larger than 1. Without loss of generality we can therefore order the eigenvalues as follows:

$$\lambda_1 = 1 \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

Assuming that a slightly stronger condition holds, namely that

$$\lambda_1 = 1 > |\lambda_2| \geq \cdots \geq |\lambda_n|,$$

then we see from (assuming as usual that P is diagonalizable)

$$x(t) = c_1(1)^t v_1 + c_2(\lambda_2)^t v_2 + \cdots + c_n(\lambda_n)^t v_n,$$

that if $c_1 \neq 0$, then

$$\lim_{t \to \infty} x(t) = c_1 v_1.$$

Although we have not shown it, it follows again from the Perron-Frobenius Theorem that the eigenvector v_1 can be chosen so that it has only non-negative components. Now, since $x(t)$ is a probability vector for all t, so must be its limit, hence it follows that $1/c_1$ must equal the sum of the components of the vector v_1, so we have that

$$\lim_{t \to \infty} x(t) = \frac{1}{\sum_{j=1}^n (v_1)_j} v_1.$$

In other words, the probability vector $x(t)$ approaches the probability vector corresponding to the eigenvector (appropriately scaled so that it is a probability vector) associated to the eigenvalue 1 as $t \to \infty$. This shows the importance of the eigenvector corresponding to the eigenvalue $\lambda_1 = 1$.

Remark 4. There are plenty of other application areas where Markov chains are used. Let us mention a few.

1. A famous example is that of Google's Page Rank, see http://en.wikipedia.org/wiki/PageRank. It is famous in part because of all the secrecy that surrounds its calculation by Google which in principle requires a tremendous computational effort. In its basic form this page rank is defined as the normalized eigenvector $(1/\sum_j(v_1)_j)v_1$ of the matrix P whose transitional probabilities p_{ij} are defined as follows:

$$p_{ij} = \begin{cases} 0, & \text{if there is no link from site } j \text{ to } i \\ \frac{1}{\text{total number of outlinks of site } j}, & \text{otherwise} \end{cases}$$

   The interpretation is that, at least asymptotically (i.e. when $t \to \infty$), a random websurfer would more likely be visiting websites whose pageranks (which are the corresponding components of v_1) are larger. Thus the pagerank establishes some kind of measure for the popularity of webpages.

2. Consider an industrial plant where products move around between workstations (assembly lines in the car industry for instance, food processing units, etc). It is clear that you can interpret the plant as a large network where you may wish to track the probability that a certain item is in one of the nodes of the network. Similarly we could model a mall in which customers move from one retailer to another, or phone networks, etc.

3. Markov chains are also used in mathematical biology. One example is that of genetic networks. Think of genes as "producers" of certain biochemical products, called proteins. They are connected in a large network and can be switched on and off, meaning that they are producing or not producing proteins. [In practice, one way genes are switched off is as follows. A certain molecule, called an inhibitor, comes and sits on the gene, essentially locking it off from the rest of the network and thereby preventing it to start producing its protein]. The state vector would therefore be a vector containing the probabilities that each of the genes is on.

Optional exercise, for mathematically inclined students: Let P be a stochastic matrix. We have shown that 1 is an eigenvalue of P, and we have claimed that it is the eigenvalue with largest modulus. Prove that claim using the following information from the -here, unproved- Perron-Frobenius Theorem from linear algebra: The eigenvalue r of the matrix P that has the largest modulus can be characterized by the following variational description:

$$r = \max\{\lambda \geq 0 \mid Ax - \lambda x \geq 0 \text{ for some } x \geq 0\},$$

where $x \geq 0$ means that all components of the vector x are non-negative.

8 Problems

For the first graded HW, do all of the following 3 problems. More problems will follow later.

1. Consider the population model again. Fix $f = 1$ and let $m \geq 0$ be arbitrary.
   (a) Show that the system is unstable and thus almost all solutions grow unbounded.
   (b) Let's modify the model to reflect death of adults. Assume that all juveniles mature between two consecutive censuses (so $f = 1$ in terms of our original model). Reproduction, and right after that death, take place at the end of one cycle, right before the census is taken. Only matures that were mature at the beginning of the cycle are capable of reproduction. Thus, the juveniles that matured in this cycle don't reproduce yet. Right after reproduction some of the adults die and a fraction s survives to the census. This census is the starting point of the next cycle. How does the original model change? (Still assume that $m \geq 0$ is arbitrary.) Give conditions for the parameters such that the system is stable, asymptotically stable and unstable respectively. Show that the condition for asymptotic stability implies that $m < 1$. Interpret this inequality.

2. Suppose that at time t you possess a certain capital $C(t)$ and some investment $I(t)$. At the start of a new investment cycle, only capital can be invested (there is no re-investment). You can invest a fraction $f \in [0, 1]$ of your capital, and thus $(1 - f)C(t)$ is the capital you keep. By the end of the financial cycle the invested fraction of the capital yields a guaranteed interest of $r\%$. However, some of the returns are delinquent, and only a fraction d of the value of the investment returns as capital to you.

   (a) Construct a model whose state consists of $C(t)$ and $I(t)$.

   (b) Let $f = 1$. Show that you will go broke if $(1 + r)d < 1$, and that you will make money if the inequality is reversed. Interpret these inequalities.

   (c) Let $f = 0.5$. How do the above inequalities change? Interpret your result.

   (d) Generalize this model by diversifying your investment. That is, assume there are n types of investments $I_1, \ldots, I_n$ each having an interest of r_i and a delinquency factor of d_i.

3. (a) Calculate the Page Rank assuming that the World Wide Web consists of 3 websites. Suppose that the websites have no self-links, and that the WWW is strongly connected (that is, given any of the websites, it is possible to reach any of the other websites following one or more links). What configuration yields the highest page rank for page 1?

   (b) Check your calculations with MATLAB, see remark below. Print a copy of you MATLAB worksheet and turn it in together with your homework.

Remark 5. MATLAB is available on all campus computers. It is software that is very friendly to matrices. When you start it up, you will see a command line >> appear. To define a matrix

$$\begin{pmatrix} 1 & 1.2 \\ 3 & 4 \end{pmatrix},$$

type

$$A = [1 \;\; 1.2 \; ; \; 3 \;\; 4]$$

after the command line and hit return. MATLAB will produce an output in the form of the desired matrix. Now that this matrix has been defined, you can calculate its eigenvector-eigenvalues pairs by typing the following:

[T,D]=eig(A)

MATLAB will return two matrices T and D, where T contains the eigenvectors and D is a diagonal matrix containing the eigenvalues on the diagonal.