Lemma used to prove Rodrigues's formula 1*

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We showed that $P_n(x)$, the Legendre polynomial of degree n, satisfies Rodrigues's formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right), \quad n = 0, 1, 2, \dots,$$

provided the following Lemma is proved.

Lemma

The polynomial $p_n(x) = \frac{d^n}{dx^n}\left((x^2-1)^n\right)$ $(n = 1, 2, \dots)$ is orthogonal to any polynomial of degree less than n.

Proof. Let $q(x)$ be a polynomial of degree less than n and consider.

$$\int_{-1}^{1} \frac{d^n}{dx^n}\left((x^2-1)^n\right) q(x) dx.$$

We need to show that this integral is 0 for all $n = 1, 2, \dots$.

Integrating by parts once yields:

$$\frac{d^{n-1}}{dx^{n-1}}\left((x^2-1)^n\right) q(x)|_{x=-1}^{x=1} - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}\left((x^2-1)^n\right) q'(x) dx.$$

Repeating this several times (in total $n-1$ times) we get that this integral equals:

$$\begin{array}{rl} \frac{d^{n-1}}{dx^{n-1}}\left((x^2-1)^n\right) q(x)|_{x=-1}^{x=1} & - \\ \frac{d^{n-2}}{dx^{n-2}}\left((x^2-1)^n\right) q'(x)|_{x=-1}^{x=1} & + \\ \frac{d^{n-3}}{dx^{n-3}}\left((x^2-1)^n\right) q''(x)|_{x=-1}^{x=1} & - \quad \dots \\ +(-1)^{n-1}\frac{d^0}{dx^0}\left((x^2-1)^n\right) q^{n-1}(x)|_{x=-1}^{x=1} & . \end{array}$$

There are only a finite number of terms because $q^{(n)}(x) = 0$ (recall that q is a polynomial of degree less than n).

We write this more compactly:

$$\sum_{k=0}^{n-1} (-1)^k \frac{d^{n-1-k}}{dx^{n-1-k}}\left((x^2-1)^n\right) q^{(k)}(x)|_{x=-1}^{x=1}$$

Notice that we'll be done if we can prove the following:

$$\frac{d^j}{dx^j}\left((x^2-1)^n\right)|_{x=-1}^{x=1} = 0, \text{ for all } n = 1, 2, \dots \text{ and } j = 0, 1, \dots, n-1.$$

The proof is by induction on n. The assertion is immediate if $n = 1$. So let's assume the assertion is true for n, and try to show it is true for $n+1$. That is, we try to show that:

$$\frac{d^j}{dx^j}\left((x^2-1)^{n+1}\right)|_{x=-1}^{x=1} = 0, \text{ for all } j = 0, 1, \dots, n.$$

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For $j = 0$ this is obvious, so we assume that $j > 0$. Then

$$\begin{aligned}
\frac{d^j}{dx^j}\left((x^2-1)^{n+1}\right)|_{x=-1}^{x=1} &= \frac{d^{j-1}}{dx^{j-1}}\left(2x(n+1)(x^2-1)^n\right)|_{x=-1}^{x+1} \\
&= 2(n+1)\sum_{k=0}^{j-1}\binom{j-1}{k}\frac{d^{j-1-k}}{dx^{j-1-k}}(x)\frac{d^k}{dx^k}\left((x^2-1)^n\right)|_{x=-1}^{x=1}
\end{aligned}$$

where we used the formula for the $(j-1)$th derivative of a product[1] The last factor in each term is zero by the induction hypothesis.

□

[1]This formula says that for two sufficiently smooth functions $f(x)$ and $g(x)$, there holds that $(fg)^n = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}g^{(k)}$. This formula is an application of the binomial formula and can be proved by induction.