Extra credit problem*

Due date: Anytime before December 1st, 2007

The Legendre polynomials are coefficients of z^n in a Taylor series expension with respect to z of a certain function $f(x,z)$:

$$f(x,z)=\sum_{n=0}^{\infty}P_n(x)z^n,\ \ |x|,|z|<1.$$

This function is called the *generating function* for the Legendre polynomials.

The purpose of this problem is to show that this function is:

$$f(x,z)=\frac{1}{\sqrt{1-2xz+z^2}}.$$

To show this, you should only use the following recurrence relation that exists between Legendre polynomials (no need to prove this recurrence relation):

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x),\ \ n=0,1,\dots$$

and the fact that $P_0(x)=1$.

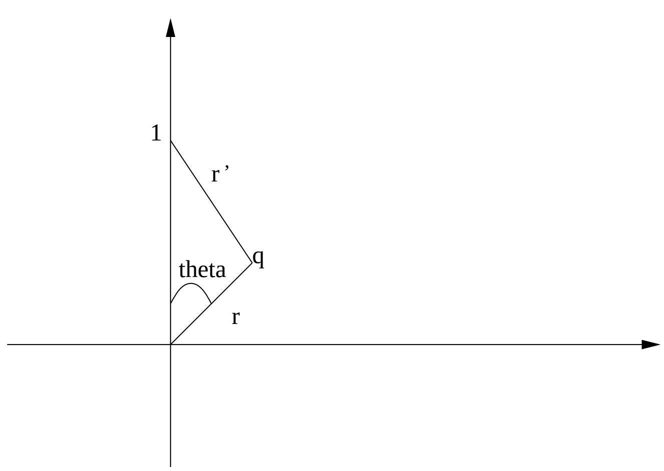
Follow the procedure outlined in problem 35 of section 8.8.

Practical use: To write down the Legendre polynomials explicitely without memorizing them, it suffices to expand the function $f(x,z)$ in a Taylor series with respect to z (while fixing x). But how to memorize $f(x,z)$? Here's a graphical way with an interesting physical interpretation.

Let an electrical charge q be located at a point with polar coordinates (r,θ) (here, θ is the angle between the radius and the y-axis) with $r<1$. In physics one shows that the potential at the point with polar coordinates $(1,0)$ on the y-axis is proportional to $1/r'$ where r' is the distance between this point and the point where the charge is:

$$\text{potential}\sim\frac{1}{r'}.$$

*MAP 4305; Instructor: Patrick De Leenheer.



Now by the law of cosines:

$$(r')^2 = 1 + r^2 - 2r\cos(\theta),$$

and thus the potential is proportional to:

$$\frac{1}{\sqrt{1 + r^2 - 2r\cos(\theta)}},$$

which is exactly $f(\cos(\theta), r)$.