Practice Exam 3: MAP 4305*

1. Calculate e^{At} for
$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

2. Verify that the region containing the origin and bounded by the line segments $L_1 : y = x+6,\ x \in [-3, 0]$, $L_2 : y = 6,\ x \in [0, 3]$, $L_3 : x = 3,\ y \in [-3.6]$, $L_4 : y = x - 6,\ x \in [0, 3]$, $L_5 : y = -6,\ x \in [-3, 0]$, and $L_6 : x = -3,\ y \in [-6, 3]$ is a trapping region for the Van der Pol oscillator:
$$\begin{aligned} \dot{x} &= y + x - x^3/3 \\ \dot{y} &= -x \end{aligned}$$
Explain why this region contains a non-constant periodic solution.

3. Using Lyapunov's direct method, establish the stability properties of the equilibrium at the origin of the system:
$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= -2x^3 - y^3 \end{aligned}$$

4. What is the solution to the following IVP:
$$\dot{x} = Ax + f(t), \quad x(0) = x_0,$$
where
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5. Is the following statement true?
$$e^A\, e^B = e^B\, e^A\,.$$
If yes, prove it; if no, provide a counterexample.

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