Practice Exam 2: MAP 2302*

1. Solve the following IVP:
$$y'' + 5y' + 4y = 0,\ \ y(0) = 1,\ \ y'(0) = 0$$
   Answer:
$$\frac{4}{3}\,\mathrm{e}^{-t} - \frac{1}{3}\,\mathrm{e}^{-4t}$$

2. Given is an RC-circuit with $R = 1$ (Ohm) and $C = 0.5$ (Farad), driven by a generator that delivers $E(t) = \sin t$ (Volts). Suppose that the initial voltage over the capacitor is 1 Volt. Determine the voltage $v_C(t)$ over the capacitor.

   Answer:
$$v_C(t) = \frac{22}{5}\,\mathrm{e}^{-2t} + \frac{4}{5}(\sin(t) - \frac{\cos(t)}{2})$$

3. What is the structure of the particular solution to:
   - $$y'' + 2y' + 2y = (t^2 + 1)\,\mathrm{e}^{-t}\cos(t)$$
     Answer:
$$t(A_2t^2 + A_1t + A_0)\,\mathrm{e}^{-t}\cos(t) + t(B_2t^2 + B_1t + B_0)\,\mathrm{e}^{-t}\sin(t)$$
   - $$y'' - 4y' + 4y = (t + 3)\,\mathrm{e}^{2t} + \sin(t) + t\,\mathrm{e}^{t}$$
     Answer:
$$t^2(A_1t + A_0)\,\mathrm{e}^{2t} + B_1\cos(t) + B_2\sin(t) + (C_1t + C_0)\,\mathrm{e}^{t}$$

4. Find the general solution on the interval $(-\pi/6, \pi/6)$ of
$$y'' + 9y = \tan(3t)$$
   Answer:
$$v_1(t)y_1(t) + v_2(t)y_2(t),$$
   where
$$y_1(t) = \cos(3t),\ \ y_2(t) = \sin(3t),\ \ v_1(t) = -\frac{1}{9}\ln|\sec(3t) + \tan(3t)| + \frac{1}{9}\sin(3t),\ \ v_2(t) = -\frac{1}{9}\cos(3t)$$

5. We discussed mass-spring-damper systems, modeled by:
$$mx'' + bx' + kx = 0,$$
   where m, b and k are positive and denote the mass, damping coefficient and spring coefficient respectively.

   Suppose we consider instead:
$$mx'' - bx' + kx = 0,$$
   where m, b and k are still assumed to be positive. Notice that we simply replaced b by $-b$. Interpret this change physically. What kind of solutions do you expect, based on this physical interpretation? Now *prove* that all non-zero solutions are unbounded, that is, show that for any solution $x(t)$ holds that
$$\lim_{t\to\infty} |x(t)| = \infty.$$
   Does this confirm your physical interpretation?

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Answer: Expect oscillations of increasing amplitude.

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\begin{aligned}
x(t) &= c_1 \,\mathrm{e}^{(b+\sqrt{b^2-4mk})t/2} + c_2 \,\mathrm{e}^{(b-\sqrt{b^2-4mk})t/2} \text{ if } b^2 - 4mk > 0 \\
x(t) &= (c_1 + c_2 t)\,\mathrm{e}^{bt/2} \text{ if } b^2 - 4mk = 0 \\
x(t) &= c_1 \,\mathrm{e}^{bt/2} \cos(\sqrt{4mk - b^2}) + c_2 \,\mathrm{e}^{bt/2} \sin(\sqrt{4mk - b^2}) \text{ if } b^2 - 4mk < 0
\end{aligned}
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In all cases, it's clear that when $c_1 c_2 \neq 0$, $|x(t)| \to +\infty$ since $b > 0$.

6. Find 2 linearly independent solutions of the system:

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\begin{aligned}
\frac{dx}{dt} &= x + y \\
\frac{dy}{dt} &= 3x - y
\end{aligned}
$$

Answer:

$$
\begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{2t}, \quad \begin{pmatrix} 1 \\ -3 \end{pmatrix} \mathrm{e}^{-2t}
$$