Homework assignment 1*

Due date: September 18

1. Determine the values of n for which the following equation is exact, and solve the equation for those values:
$$(x + y\,\mathrm{e}^{2xy})dx + nx\,\mathrm{e}^{2xy}\,dy = 0$$

2. **Bernoulli's equation**

   Bernoulli 's equation:
$$\frac{dy}{dx} + p(x)y = q(x)y^n$$
   is linear if (and only if) n equals 0 or 1. Nevertheless, show that the transformation
$$z = y^{1-n}$$
   yields a linear equation. Use this transformation to solve:
$$y' + xy = xy^4$$

3. **The catenary (aka the shape of a hanging cable)**

   Imagine a power cable whose ends are attached to 2 poles of the same height. The poles are stuck in the ground, which we assume forms a nice flat and horizontal surface. The weight of the cable is uniform and we denote the weight of the cable per unit length by w. The horizontal tension in the midpoint of the cable (where the cable is horizontal) is denoted by T. Introduce a Cartesian coordinate system (x, y) with origin on the ground right below the midpoint of the cable. The coordinate of this midpoint is denoted by $(0, y_0)$. We want to find the height $y(x)$ of each point of the cable. It can be shown (don't do this!) that y satisfies the following differential equation:
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{w}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$
   Show that the height of the cable is given by:
$$y(x) = \frac{T}{w}\left(\cosh\left(\frac{w}{T}x\right) - 1\right) + y_0$$
   A nice example of an inverted catenary is the Gateway Arch in St. Louis.

4. **Families of orthogonal curves**

   Given a family of curves $\mathcal{F}$, it is often important to determine another family of curves $\mathcal{F}'$ with the property that all curves of $\mathcal{F}'$ are orthogonal (or perpendicular) to all curves of $\mathcal{F}$. For instance, from physics we know that the motion of an electrical charge is perpendicular to the equipotential curves (i.e. curves consisting of points having the same electrical potential), and thus knowing the equipotential curves implies that we have an idea of the paths that electrical charges will follow. Of course, we'd like to determine these orthogonal curves more precisely.

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Let us consider a simple example. Take the family of circles centered at the origin. Each member of this family is determined by

$$x^2 + y^2 = r^2,$$

for some $r \in \mathbb{R}$. The slope in an arbitrary point of the curve to this curves is of course given by dy/dx. In this case, by differentiating the above equation with respect to x, and solving for dy/dx, we find:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Here is the key property that will determine the family of orthogonal curves: The slope of each member of the orthogonal family is the *negative reciprocal* of the slope of the original family. In other words, for each member of the orthogonal family the slope is given by:

$$\frac{dy}{dx} = -\left(-\frac{1}{x/y}\right) = \frac{y}{x}$$

Solving this differential equation yields:

$$y(x) = kx,$$

where k is arbitrary. This is a family of straight lines through the origin. Of course, this is what we expected all along, based on our geometric understanding of the problem.

- Verify the above calculations.
- Repeat this reasoning to determine the family which is orthogonal to the family given by:

  $$y = cx^2,$$

  where $c \in \mathbb{R}$. Sketch both families.

5. **The Riccati equation**

Consider the following pair of differential equations:

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy, \end{aligned}$$

where a, b, c and d are real parameters. This is an example of a *system* of differential equations, a topic to be studied in more detail later in the course. We say that a pair of functions $(x(t), y(t))$ with t in some interval, is a solution of this system if both functions satisfy the above equations. A solution can thus be thought of as a parameterized curve in the (x, y) plane. Suppose that we are not interested in determining the precise solution, but only in some angular information about the point traveling along the solution curve. More precisely, suppose we would like to know the tangent of the angle between the x-axis and the vector connecting the origin with the point $(x(t), y(t))$ on the solution curve. Clearly, this tangent is given by

$$z = \frac{y}{x}$$

- Show that the variable z satisfies the so-called Riccati equation:

  $$\dot{z} = c + (d - a)z - bz^2$$

- Classify the Riccati equation: Is it linear or nonlinear, and what is its order?
- Suppose that $z_p(t)$ is a particular solution of the Riccati equation. Then the general solution z of the Riccati equation can be written as

  $$z = z_p + \frac{1}{q}$$

  Show that the variable q satisfies a *linear differential equation*.
- Consider the special case where $a = d$ arbitrary, $c = 1$ and $b = -1$. Show that a particular solution of the Riccati equation is $\arctan x$. Then find the general solution z, by solving the linear differential equation for q.